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Algebraic Geometry 2

Sheaves and Cohomology

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ABSTRACT. This is the second of three books by the author aimed at introducing the reader to Grothendieck's scheme theory as a method of studying algebraic geometry. This book contains definitions and results related to coherent schemes, proper and projective morphisms, and cohomology of sheaves on schemes. As in the first book, the author includes many examples and problems illustrating the topics discussed in the main text.

The book is aimed at graduate and upper-level undergraduate students who want to learn modern algebraic geometry.

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Contents

CHAPTER 4

Coherent Sheaves

In this chapter we will discuss the most important concept in algebraic geometry: coherent sheaves. Kiyoshi Oka introduced the concept of an ideal with indeterminate domains (that is, the stalk $\mathcal{O}_{X,x}$ at x of a sheaf $\mathcal{O}_X$ of holomorphic functions) and discovered the important properties of the ideal of indeterminate domains. H. Cartan recognized that Oka's work essentially coincided with the notion of Leray's sheaf. Consequently, by introducing the concept of a coherent sheaf, Cartan expressed Oka's results as the coherency of the sheaf of holomorphic functions. Cartan and J.-P. Serre reinterpreted the main results in the theory of holomorphic functions of several complex variables in terms of coherent sheaves. Grothendieck's scheme theory is the ultimate result of Serre's plan. These historical events indicate how important coherent sheaves are. For the applications of coherent sheaves to schemes, we find it more convenient to generalize the notion of a coherent sheaf to that of a quasicoherent sheaf. Following Grothendieck, we will begin with the theory of quasicoherent sheaves. Note that sheaves which are neither coherent nor quasicoherent play an important role in algebraic geometry.

We have briefly described the theory of sheaves. In this chapter we will establish the fundamental properties of sheaves on schemes in detail. Sheaf theory requires one entire book for its full treatment. Consequently, books on algebraic geometry cover sheaf theory only by giving necessary definitions, and then proceed to the next topic. Our intention is to describe sheaves in as much detail as possible. It is often the case that a lack of understanding of sheaf theory causes students to have difficulties grasping algebraic geometry. In order to understand sheaf theory, it is important to witness sheaves in action. For this purpose, we will provide many examples.

The sheaf induced by an R-module M over $\operatorname{Spec} R$ is denoted as either $\widetilde{M}$ or $(M)^{\sim}$. When the description for M is long, we will use the latter notation.

4.1. Exact Sequence of Sheaves

A homomorphism of sheaves has been described in §2.3(a). We will discuss it fully in this section. We will define the kernel, the image, and the cokernel of a sheaf homomorphism, and show that the notion of sheaves naturally generalizes that of additive groups (abelian groups). All of our sheaves or presheaves are assumed to be sheaves or presheaves of additive groups.

(a) **Sheafification of Presheaves**

We will review the construction of a sheaf of a presheaf $\mathcal{G}$ over a topological space X (see Exercise 2.5). Define the stalk $\mathcal{G}_x$ of $\mathcal{G}$ at $x \in X$ as follows:

$$\mathcal{G}_x = \varinjlim_{x \in U} \mathcal{G}(U).$$

The right-hand side is the inductive limit (direct limit) over $\mathcal{U}_x$ of all the open sets containing x where the order $U < V$ is defined by $V \subset U$. Let ${}^a\mathcal{G}(U)$ be the collection (totality) of maps s from U to $\bigcup_{x \in U} \mathcal{G}_x$ satisfying the following:

(1) $s(x) \in \mathcal{G}_x$ for $x \in U$.

(2) For each $x \in U$, one can choose an open set $V \subset U$ containing x, and $t \in \mathcal{G}(V)$ so that the germ $t_y \in \mathcal{G}_y$ of t at an arbitrary point y in V coincides with $s(y)$. Namely, define

(4.1)
$$ {}^a\mathcal{G}(U) = \left\{ \{s(x)\} \in \prod_{x \in U} \mathcal{G}_x \;\middle|\; \begin{array}{l} \text{an open neighborhood } V \subset U \text{ and} \\ t \in \mathcal{G}(U) \text{ can be chosen so that} \\ t_y = s(y),\ y \in V \end{array} \right\}. $$

The restriction map $\rho_{V,U} : {}^a\mathcal{G}(U) \to {}^a\mathcal{G}(V)$ is defined by restricting $\{s(x)\}_{x \in U}$ to $\{s(y)\}_{y \in V}$. Then ${}^a\mathcal{G}$ is a sheaf of additive groups over X.

PROBLEM 1. Prove that ${}^a\mathcal{G}$ is a sheaf of additive groups.

PROBLEM 2. Prove that the sheaf ${}^a\mathcal{G}$ coincides with the sheaf $\widetilde{\mathcal{G}}$ as defined in Exercise 2.5.

For presheaves $\mathcal{G}$ and $\mathcal{H}$ of additive groups over a topological space X, a *homomorphism* $\varphi : \mathcal{G} \to \mathcal{H}$ is defined as follows. For each open set U in X, a homomorphism $\varphi_U : \mathcal{G}(U) \to \mathcal{H}(U)$ is defined satisfying the compatibility condition, i.e., for open sets $V \subset U$ the diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\varphi_U} & \mathcal{H}(U) \\ {\scriptstyle \rho^{\mathcal{G}}_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{H}}_{V,U}} \\ \mathcal{G}(V) & \xrightarrow{\varphi_V} & \mathcal{H}(V) \end{array}$$

commutes. Namely, φ is a natural transformation from $\mathcal{G}$ to $\mathcal{H}$ (see §3.1). In particular, when either $\mathcal{G}$ or $\mathcal{H}$ is a sheaf, we can consider a presheaf homomorphism from $\mathcal{G}$ to $\mathcal{H}$. Then a sheaf homomorphism coincides with a presheaf homomorphism. Let $\mathrm{Hom}_{\mathrm{sheaf}}(\mathcal{G}, \mathcal{H})$ be the totality of homomorphisms from a sheaf $\mathcal{G}$ to a sheaf $\mathcal{H}$, and let $\mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H})$ be the totality of homomorphisms from a presheaf $\mathcal{G}$ to a presheaf $\mathcal{H}$. That is, we have

$$\mathrm{Hom}_{\mathrm{sheaf}}(\mathcal{G}, \mathcal{H}) = \mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H}).$$

Note also that $\mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H})$ and $\mathrm{Hom}_{\mathrm{sheaf}}(\mathcal{G}, \mathcal{H})$ are additive groups. Namely, for $\varphi : \mathcal{G} \to \mathcal{H}$ and $\psi : \mathcal{G} \to \mathcal{H}$, and for each open set U of X, define $\varphi + \psi$ as follows. For $s \in \mathcal{G}(U)$,

$$(\varphi + \psi)_U(s) \equiv \varphi_U(s) + \psi_U(s).$$

The zero element is the zero map 0, i.e., for each open set U and $s \in \mathcal{G}(U)$ the zero map is a homomorphism satisfying

$$O_U(s) = 0.$$

A presheaf (or sheaf) homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$ is said to be an *isomorphism* if for each open set U of X, φ_U is an isomorphism. Then presheaves (or sheaves) $\mathcal{G}$ and $\mathcal{H}$ are said to be *isomorphic*.

For a presheaf $\mathcal{G}$ over a topological space X, we constructed a sheaf ${}^a\mathcal{G}$. Then, for an open set U in X, one can define a homomorphism of additive groups

$$\begin{aligned} \theta_U : \mathcal{G}(U) &\to {}^a\mathcal{G}(U), \\ t &\mapsto \{t_x\}_{x \in U}, \end{aligned} \tag{4.2}$$

where t_x is the germ of $t \in \mathcal{G}(U)$ at x. From the definition (4.1) and the definition of the restriction map of ${}^a\mathcal{G}$, it is clear that (4.2) defines a presheaf homomorphism $\theta : \mathcal{G} \to {}^a\mathcal{G}$.

The following proposition characterizes the sheaf ${}^a\mathcal{G}$ and the homomorphism $\theta : \mathcal{G} \to {}^a\mathcal{G}$.

PROPOSITION 4.1. (i) *For presheaves $\mathcal{G}$ and $\mathcal{H}$ of additive groups over a topological space X, the map determined by the sheaf ${}^a\mathcal{G}$ of additive groups and the homomorphism of presheaves in* (4.2)

$$\begin{aligned} &\mathrm{Hom}_{\mathrm{sheaf}}({}^a\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H}), \\ &\qquad\qquad \varphi \mapsto \varphi \circ \theta, \end{aligned} \tag{4.3}$$

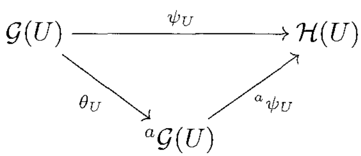
is an isomorphism of additive groups. Conversely, for a presheaf $\mathcal{G}$ of additive groups, the sheaf ${}^a\mathcal{G}$ and the homomorphism of presheaves $\theta : \mathcal{G} \to {}^a\mathcal{G}$ are determined uniquely up to isomorphisms.

(ii) *When $\mathcal{G}$ is a sheaf, we have ${}^a\mathcal{G} = \mathcal{G}$.*

PROOF. By the definition of the map in (4.3), the map is a homomorphism of additive groups. For ${}^a\mathcal{G}$ as constructed above, we define a map

$$\mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_{\mathrm{sheaf}}({}^a\mathcal{G}, \mathcal{H}). \tag{4.4}$$

For a given homomorphism of presheaves $\psi : \mathcal{G} \to \mathcal{H}$, we will show that for an open set U, one can construct a homomorphism of additive groups ${}^a\psi_U : {}^a\mathcal{G}(U) \to \mathcal{H}(U)$ making the diagram



commutative. From (4.1), for $s = \{s(x)\} \in {}^a\mathcal{G}(U)$, one can find an open covering $\{V_\lambda\}_{\lambda\in\Lambda}$ of U and $t_\lambda \in \mathcal{G}(V_\lambda)$ so that $s(y) \equiv t_{\lambda_y}$, $y \in V_\lambda$. Let $\tilde{t}_\lambda = \varphi_{V_\lambda}(t_\lambda)$. Then we have $\tilde{t}_\lambda \in \mathcal{H}(V_\lambda)$. For $V_{\lambda\mu} = V_\lambda \cap V_\mu \neq \varnothing$, we have $\rho_{V_{\lambda\mu},V_\lambda}(\tilde{t}_\lambda) = \rho_{V_{\lambda\mu},V_\mu}(\tilde{t}_\mu)$ since $\rho_{V_{\lambda\mu},V_\mu}(t_\lambda) = \rho_{V_{\lambda\mu},V_\mu}(t_\mu)$. Since $\mathcal{H}$ is a sheaf, property (F2) in §2.2 implies that $\tilde{t} \in \mathcal{H}(U)$ exists satisfying $\rho_{V_\lambda,U}(\tilde{t}) = t_\lambda$, $\lambda \in \Lambda$. Furthermore, from (F1) such a $\tilde{t}$ is uniquely determined. Then define ${}^a\psi_U(s) = \tilde{t}$. For $t \in \mathcal{G}(U)$, let $s = \{t_x\} \in {}^a\mathcal{G}(U)$. We have $\theta_U(t) = s$ and $\tilde{t} = \psi_U(t)$. Namely, we obtain ${}^a\psi_U \circ \theta_U = \psi_U$. Consequently, the above diagram is commutative. Therefore, ${}^a\psi \circ \theta = \psi$ implies that the map (4.3) is surjective.

Conversely, suppose $\psi = \varphi \circ \theta, \varphi \in \mathrm{Hom}_{\mathrm{sheaf}}({}^a\mathcal{G}, \mathcal{H})$. Then from the construction of ${}^a\psi$ in the above we get ${}^a\psi = \varphi$. Hence, the map (4.3) is injective, and so it is an isomorphism. (The map (4.4) is the inverse map of (4.3).)

Assume that for a sheaf $\mathcal{F}$ and for a presheaf homomorphism $\eta : \mathcal{G} \to \mathcal{F}$, the map

$$\mathrm{Hom}_{\mathrm{sheaf}}(\mathcal{F}, \mathcal{H}) \to \mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H}),$$
$$\varphi \mapsto \varphi \circ \eta,$$

is an isomorphism, where $\mathcal{H}$ is an arbitrary sheaf. In particular, for $\mathcal{H} = {}^a\mathcal{G}$ and $\theta : \mathcal{G} \to {}^a\mathcal{G}$, a sheaf homomorphism $\varphi : \mathcal{F} \to {}^a\mathcal{G}$ is uniquely determined to satisfy $\theta = \varphi \circ \eta$. In (4.3) let $\mathcal{H} = \mathcal{F}$. Then a sheaf homomorphism $\psi : {}^a\mathcal{G} \to \mathcal{F}$ is uniquely determined to satisfy $\eta = \psi \circ \theta$. We have $\theta = \varphi \circ \eta = \theta \circ (\psi \circ \theta) = (\varphi \circ \psi) \circ \theta$, which implies that, for a sheaf homomorphism $\varphi \circ \psi : {}^a\mathcal{G} \to {}^a\mathcal{G}$, if $\mathcal{H} = {}^a\mathcal{G}$ in (4.3), $\varphi \circ \psi$ corresponds to $\theta \in \mathrm{Hom}_{\mathrm{presheaf}}(\mathcal{G}, \mathcal{H})$. On the other hand, $\mathrm{id}_{{}^a\mathcal{G}}$ also corresponds to θ in (4.3). Since (4.3) is an isomorphism, we have $\varphi \circ \psi = \mathrm{id}_{{}^a\mathcal{G}}$. Furthermore, $\eta = \psi \circ \theta = \psi \circ (\varphi \circ \eta) = (\psi \circ \varphi) \circ \eta$ implies that the isomorphism (4.5) provides $\psi \circ \varphi = \mathrm{id}_{\mathcal{F}}$ as well. Therefore, $\psi : {}^a\mathcal{G} \to \mathcal{F}$ is an isomorphism, and $\eta = \psi \circ \theta$ implies that $({}^a\mathcal{G}, \theta)$ is uniquely determined up to an isomorphism to satisfy (4.3).

(ii) If $\mathcal{G}$ is a sheaf, then by properties (F1) and (F2), the definition of ${}^a\mathcal{G}$ in (4.1) implies ${}^a\mathcal{G}(U) = \mathcal{G}(U)$. □

As was mentioned in Exercise **2.5**, ${}^a\mathcal{G}$ is said to be *sheafification* of a presheaf $\mathcal{G}$. The above proposition characterizes the sheafification by the universal mapping property.

EXAMPLE 4.2. For a presheaf $\mathcal{G}$ and its sheafification ${}^a\mathcal{G}$ over a topological space X, their stalks $\mathcal{G}_x$ and ${}^a\mathcal{G}_x$ at x coincide.

PROOF. For an open set U containing $x \in X$ and $s \in {}^a\mathcal{G}(U)$, one can find $x \in V \subset U$ and $t \in \mathcal{G}(V)$ so that $\rho_{V,U}(s) = \theta_U(t)$. Hence, we get $s_x = t_x$. □

(b) **Kernels and Cokernels of Sheaf Homomorphisms**

For a given homomorphism of sheaves of additive groups $\varphi : \mathcal{G} \to \mathcal{H}$ over a topological space X, we will define the kernel and the cokernel of φ as sheaves of additive groups. We will begin with the kernel, which is simpler than the cokernel.

EXERCISE 4.3. Let $\varphi : \mathcal{G} \to \mathcal{H}$ be a sheaf homomorphism over a topological space X. For an open set U of X, define

$$\mathcal{F}(U) = \{s \in \mathcal{G}(U) | \varphi_U(s) = 0\}. \tag{4.5}$$

Then $\mathcal{F}$ is a sheaf of additive groups over X.

PROOF. For open sets $V \subset U$, let $\rho^{\mathcal{G}}_{V,U}$ and $\rho^{\mathcal{H}}_{V,U}$ be the restriction maps of sheaves $\mathcal{G}$ and $\mathcal{H}$, respectively. From the definition of a sheaf homomorphism, we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{G}(U) & \xrightarrow{\varphi_U} & \mathcal{H}(U) \\ {\scriptstyle \rho^{\mathcal{G}}_{V,U}}\downarrow & & \downarrow{\scriptstyle \rho^{\mathcal{H}}_{V,U}} \\ \mathcal{G}(V) & \xrightarrow{\varphi_V} & \mathcal{H}(V) \end{array}$$

Then we have $\rho^{\mathcal{G}}_{V,U}(\mathcal{F}(U)) \subset \mathcal{F}(V)$. Define the restriction map $\rho_{V,U}$ for $\mathcal{F}$ as the restriction of $\rho^{\mathcal{G}}_{V,U}$ to $\mathcal{F}(U)$. Then $\mathcal{F}$ is a presheaf of additive groups over X.

We will show that $\mathcal{F}$ satisfies the sheaf properties (F1) and (F2). Since $\mathcal{G}$ is a sheaf, $\mathcal{F}$ clearly satisfies (F1). For an open set U in X, let $\{U_j\}_{j\in I}$ be an open covering of U. Suppose that for $U_{jk} = U_j \cap U_k \neq \varnothing$ we have $\rho_{U_{jk},U_j}(s_j) = \rho_{U_{jk},U_k}(s_k)$ for $s_j \in \mathcal{F}(U_j)$, $j \in I$. By regarding $s_j \in \mathcal{G}(U_j)$, there is $s \in \mathcal{G}(U)$ such that $\rho^{\mathcal{G}}_{U_j,U}(s) = s_j$, $j \in I$. Let $t = \varphi_U(s)$ and $t_j = \rho^{\mathcal{H}}_{U_j,U}(t)$. Then $t_j = \varphi_{U_j}(\rho^{\mathcal{G}}_{U_j,U}(s)) = \varphi_{U_j}(s_j) = 0$. Since $\mathcal{H}$ is a sheaf, we get $t = 0$. Therefore $s \in \mathcal{F}(U)$, satisfying (F2). □

The sheaf $\mathcal{F}$ defined by (4.5) is said to be the *kernel* of $\varphi : \mathcal{G} \to \mathcal{H}$ and is denoted by $\operatorname{Ker}\varphi$.

In general, for sheaves $\mathcal{F}$ and $\mathcal{G}$, if $\mathcal{F}(U)$ is an additive subgroup of $\mathcal{G}(U)$, and if the restriction map $\rho_{V,U} : \mathcal{F}(U) \to \mathcal{F}(V)$ is obtained by restricting $\rho^{\mathcal{G}}_{V,U}$ to $\mathcal{F}(U)$, then $\mathcal{F}$ is said to be a *subsheaf* of $\mathcal{G}$. The above $\operatorname{Ker}\varphi$ is an example of a subsheaf of $\mathcal{G}$. For a sheaf homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$, we will see whether

$$\mathcal{I}(U) = \operatorname{Im}\varphi_U = \{\varphi_U(s) \in \mathcal{H}(U) | s \in \mathcal{G}(U)\}$$

is a subsheaf of $\mathcal{H}$ or not, where U is an open set of X. For $V \subset U$ and for $t \in \mathcal{I}(U)$, there is an element $s \in \mathcal{G}(U)$ satisfying $t = \varphi_U(s)$. Hence, we get

$$\rho^{\mathcal{H}}_{V,U}(t) = \rho^{\mathcal{H}}_{V,U}(\varphi_U(s)) = \varphi_V(\rho^{\mathcal{G}}_{V,U}(s)) \in \mathcal{I}(V).$$

Namely, $\mathcal{I}$ becomes a presheaf whose restriction map is obtained from that of $\mathcal{H}$. We will examine (F1) and (F2) for $\mathcal{I}$. For an open covering $\{U_j\}_{j\in J}$ of U, if $t \in \mathcal{I}(U)$ satisfies $\rho_{U_i,U}(t) = 0$, then as an element of $\mathcal{H}(U)$ we have $t = 0$. Hence, as an element of $\mathcal{I}(U)$, we get $t = 0$. Namely, $\mathcal{I}$ satisfies (F1).

Suppose, for an open covering $\{U_j\}_{j\in J}$ of U, that $t_j \in \mathcal{I}(U)$, $j \in J$, satisfy $\rho_{U_{jk},U_j}(t_j) = \rho_{U_{jk},U_k}(t_k)$, where $U_{jk} = U_j \cap U_k \neq \varnothing$. Since $\mathcal{H}$ is a sheaf, by regarding $t_j \in \mathcal{H}(U_j)$, $j \in J$, there is $t \in \mathcal{H}(U)$ satisfying $\rho^{\mathcal{H}}_{U_j,U}(t) = t_j$. Is $t \in \mathcal{I}(U)$? Namely, we ask whether there exists $s \in \mathcal{G}(U)$ satisfying $\varphi_U(s) = t$.

Since $t_j \in \mathcal{I}(U_j)$, there exists $s_j \in \mathcal{G}(U_j)$ such that $t_j = \varphi_{U_j}(s_j)$. Notice that s_j is not uniquely determined. If $t_j = \varphi_{U_j}(s'_j)$, $s'_j \in \mathcal{G}(U_j)$, then $\varphi_{U_j}(s_j - s'_j) = 0$, i.e., $s_j - s'_j \in (\operatorname{Ker}\varphi)(U_j)$. That is, for any $u_j \in (\operatorname{Ker}\varphi)(U_j)$, we have $t_j = \varphi_{U_j}(s_j) = \varphi_{U_j}(s_j + u_j)$. For each $j \in J$, choose one element $s_j \in \mathcal{G}(U_j)$ satisfying $t_j = \varphi_{U_j}(s_j)$. Then we would like to choose $u_j \in (\operatorname{Ker}\varphi)(U_j)$ so that $\tilde{s}_j = s_j + u_j$ may satisfy $\rho^{\mathcal{G}}_{U_{jk},U_j}(\tilde{s}_j) = \rho^{\mathcal{G}}_{U_{jk},U_k}(\tilde{s}_k)$ for $U_{jk} = U_j \cap U_k \neq \varnothing$. If such a choice is possible, then there is $\tilde{s} \in \mathcal{G}(U)$ such that $\rho^{\mathcal{G}}_{U_j,U}(\tilde{s}) = \tilde{s}_j$, and $t_j = \varphi_{U_j}(\tilde{s}_j)$ would give $t = \varphi_U(\tilde{s})$. Thus, the question is if such $\{u_j\}_{j\in J}$ exist or not. This problem depends upon the sheaf $\operatorname{Ker}\varphi$. Let us explain this situation. For $U_{jk} = U_j \cap U_k \neq \varnothing$, let

$$s_{jk} = \rho_{U_{jk},U_k}(s_k) - \rho_{U_{jk},U_j}(s_j) \in \mathcal{G}(U_{jk}). \tag{4.6}$$

Then $t_j = \varphi_{U_j}(s_j), t_k = \varphi_{U_k}(s_k)$ and $\rho^{\mathcal{H}}_{U_{jk},U_j}(t_j) = \rho^{\mathcal{H}}_{U_{jk}.U_k}(t_k)$ imply $s_{jk} \in (\operatorname{Ker}\varphi)(U_{jk})$. Therefore, for $\{s_{jk}\}$ the question becomes the following. Are there $u_j \in (\operatorname{Ker}\varphi)(U_j)$, $j \in J$, such that we get

$$s_{jk} = u_k - u_j, \quad U_{jk} \neq \varnothing?$$

Namely, it is a problem about the sheaf $\operatorname{Ker}\varphi$. We have that the s_{jk} satisfy

$$s_{kj} = -s_{jk}$$

and for $U_{ijk} = U_i \cap U_j \cap U_k \neq \varnothing$

$$\rho_{U_{ijk},U_{ij}}(s_{ij}) + \rho_{U_{ijk},U_{jk}}(s_{jk}) + \rho_{U_{ijk},U_{ki}}(s_{ki}) = 0. \tag{4.7}$$

Then $\{s_{jk}\}$ is said to be a *one-cocycle*, which will play an important role for *sheaf cohomology*. We will return to sheaf cohomology in Chapter 6. The notion of the image of a sheaf homomorphism has brought us into sheaf cohomology, indicating that sheaf cohomology theory is a naturally needed concept.

A natural example of a presheaf which does not satisfy (F2) is a cokernel. The sheaf associated to the presheaf $\mathcal{I}$ is denoted as $\operatorname{Im}\varphi$, and $\operatorname{Im}\varphi$ is said to be the *image* of $\varphi : \mathcal{G} \to \mathcal{H}$.

PROBLEM 3. Prove that the s_{jk}, defined by (4.6), satisfy (4.7).

We will define the cokernel of a homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$. For each open set U of X, define

$$\mathcal{C}(U) = \operatorname{Coker} \varphi_U = \mathcal{H}(U)/\operatorname{Im} \varphi_U, \tag{4.8}$$

which is clearly a presheaf of additive groups. The restriction map $\rho_{V,U}$ is induced naturally from the restriction map $\rho^{\mathcal{H}}_{V,U}$, i.e.,

$$\rho_{V,U}(t \bmod \varphi_U(\mathcal{G}(U))) = \rho^{\mathcal{H}}_{V,U}(t) \bmod \varphi_V(\mathcal{G}(V)).$$

The following example will show that $\mathcal{C}$ need not be a sheaf.

EXAMPLE 4.4. Let $\mathbb{P}^1_k$ be the one-dimensional projective space over a field k (see Example 2.31). By Definition (c) of §2.3, $\mathbb{P}^1_k = \operatorname{Proj} k[x_0, x_1]$. Let $x = x_1/x_0$ and $y = x_0/x_1$. Then affine lines

$$U_0 = \operatorname{Spec} k[x] \quad \text{and} \quad U_1 = \operatorname{Spec} k[y]$$

form an open covering for $\mathbb{P}^1_k$. The points, denoted as $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$, are determined by homogeneous ideals $\mathfrak{p}_0 = (x_1)$ and $\mathfrak{p}_\infty(x_0)$ in $k[x_0, x_1]$, respectively. Then $\mathfrak{p}_0$ is contained in U_0, which is the point on $\operatorname{Spec} k[x]$ determined by the ideal (x) of $k[x]$. Namely, $\mathfrak{p}_0$ is the origin of the affine line $\operatorname{Spec} k[x]$. On the other hand, we have $\mathfrak{p}_\infty \in U_\infty$, which is the origin of the affine line $\operatorname{Spec} k[y]$. Define a subsheaf $\mathcal{J}$ of $\mathcal{O}_{\mathbb{P}^1_k}$ as follows:

$$\mathcal{J}(U) = \begin{cases} \mathcal{O}_{\mathbb{P}^1_k}(U) \text{ if } \mathfrak{p}_0, \mathfrak{p}_\infty \notin U, \\ \left\{ s \in \mathcal{O}_{\mathbb{P}^1_k}(U) \;\middle|\; \begin{array}{l} s(\mathfrak{p}_0) = 0 \text{ for } \mathfrak{p}_0 \in U, \\ s(\mathfrak{p}_\infty) = 0 \text{ for } \mathfrak{p}_\infty \in U \end{array} \right\}. \end{cases}$$

It is clear that $\mathcal{J}$ is a subsheaf of $\mathcal{O}_{\mathbb{P}^1_k}$. A natural homomorphism $\iota : \mathcal{J} \to \mathcal{O}_{\mathbb{P}^1_k}$ is induced by $\mathcal{J}(U) \subset \mathcal{O}_{\mathbb{P}^1_k}(U)$. Notice also that for $U_0 = \operatorname{Spec} k[x]$ we get

$$\mathcal{O}_{\mathbb{P}^1_k}(U_0) = k[x] \quad \text{and} \quad \mathcal{J}(U_0) = (x),$$

and for $U_1 = \operatorname{Spec} k[y]$, we get

$$\mathcal{O}_{\mathbb{P}^1_k}(U_1) = k[y] \quad \text{and} \quad \mathcal{J}(U_1) = (y).$$

Next we will show that

$$\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = \mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k) = k.$$

For $f \in \mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k)$, let $F = \rho_{U_0,\mathbb{P}^1_k}(f)$ and $G = \rho_{U_1,\mathbb{P}^1_k}(f)$. Then $f \in k[x]$ and $G \in k[y]$. We also get $U_{01} = U_0 \cap U_1 = \operatorname{Spec} k[x, \frac{1}{x}]$ and $\rho_{U_{01},U_0}(F) = \rho_{U_{01}U_1}(G)$, where in U_{01}, y (more precisely $\rho_{U_{01},U_1}(y)$) equals $1/x$. Hence $F(x) = G(1/x)$. Since F and G are polynomials in x and y, respectively, F and G are the same constant. Namely, we get $f \in k$. Unless $f = 0$, $f \in k$ does not become 0 at $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$. Therefore,

$$\mathcal{J}(\mathbb{P}^1_k) = 0.$$

Under the above preparation, we will study the presheaf $\mathcal{C}$ determined by the homomorphism $\iota : \mathcal{J} \to \mathcal{O}_{\mathbb{P}^1_k}$. Since for each open set U of $\mathbb{P}^1_k$ we have

$$\mathcal{C}(U) = \operatorname{Coker} \iota_U = \mathcal{O}_{\mathbb{P}^1_k}(U)/\mathcal{J}(U),$$

in particular we get

$$\mathcal{C}(U_0) = k[x]/(x) \xrightarrow{\sim} k \quad \text{and} \quad \mathcal{C}(U_1) = k[y]/(y) \xrightarrow{\sim} k.$$

That is, we regard $\mathcal{C}(U_0) = k$ and $\mathcal{C}(U_1) = k$. Since $\mathfrak{p}_0 \notin U_{01}$ and $\mathfrak{p}_\infty \notin U_{01}$, we have $\mathcal{J}(U_{01}) = \mathcal{O}_{\mathbb{P}^1_k}(U_{01})$. Consequently,

$$\mathcal{C}(U_{01}) = 0.$$

For the open covering $\{U_0, U_1\}$ of $\mathbb{P}^1_k$, let a and b be arbitrary elements in $k = \mathcal{C}(U_0)$ and $k = \mathcal{C}(U_1)$, respectively. Then

$$\rho_{U_{01},U_0}(a) = 0 \quad \text{and} \quad \rho_{U_{01},U_1}(b) = 0.$$

In particular, for the case where $a \neq b$, since

$$\mathcal{C}(\mathbb{P}^1_k) = \mathcal{O}_{\mathbb{P}^1_k}(\mathbb{P}^1_k)/\mathcal{J}(\mathbb{P}^1_k) = k,$$

one cannot find $f \in \mathcal{C}(\mathbb{P}^1_k)$ to satisfy $\rho_{U_0,\mathbb{P}^1_k}(f) = a$ and $\rho_{U_1,\mathbb{P}^1_k}(f) = b$. Therefore, the presheaf $\mathcal{C}$ does not satisfy (F2), i.e., $\mathcal{C}$ is not a sheaf. □

PROBLEM 4. For an n-dimensional projective space

$$\mathbb{P}^n_k = \operatorname{Proj} k[x_0, x_1, \ldots, x_n]$$

(see Example 2.32), show that

$$\Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}) = k.$$

Thus, for a sheaf homomorphism $\varphi : \mathcal{G} \to \mathcal{H}$, we have the associated sheaf $\operatorname{Coker} \varphi$, i.e., the sheafification of the presheaf $\mathcal{C}$ defined as in (4.8). The sheaf $\operatorname{Coker} \varphi$ is said to be the *cokernel* of φ. As was explained, the image and cokernel of a sheaf homomorphism need to be sheafified. However, at stalks those associated sheaves are as simple as additive groups.

THEOREM 4.5. *Let* $\varphi_x : \mathcal{G}_x \to \mathcal{H}_x$ *be the induced homomorphism of stalks at* x *by a homomorphism* $\varphi : \mathcal{G} \to \mathcal{H}$ *of sheaves of additive groups. Then the stalks* $(\operatorname{Ker}\varphi)_x, (\operatorname{Im}\varphi)_x, (\operatorname{Coker}\varphi)_x$ *of* $\operatorname{Ker}\varphi, \operatorname{Im}\varphi, \operatorname{Coker}\varphi$*, respectively, at* x *coincide with the kernel, image, and cokernel of an additive group homomorphism* φ_x*. Namely, we have*

$$\begin{aligned}(\operatorname{Ker}\varphi)_x &= \operatorname{Ker}\varphi_x,\\ (\operatorname{Im}\varphi)_x &= \operatorname{Im}\varphi_x = \varphi_x(\mathcal{G}_x),\\ (\operatorname{Coker}\varphi)_x &= \operatorname{Coker}\varphi_x = \mathcal{H}_x/\varphi_x(\mathcal{G}_x).\end{aligned}$$

PROOF. For an open set U of X, let

$$\begin{aligned}&\mathcal{F}(U) = \operatorname{Ker}\{\varphi_U : \mathcal{G}(U) \to \mathcal{H}(U)\},\\ &\mathcal{I}(U) = \varphi_V(\mathcal{G}(U)),\\ &\mathcal{C}(U) = \mathcal{H}(U)/\mathcal{I}(U).\end{aligned}$$

Then we have the following exact sequences of additive groups:

$$\begin{aligned}&0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{I}(U) \to 0,\\ &0 \to \mathcal{I}(U) \to \mathcal{H}(U) \to \mathcal{C}(U) \to 0.\end{aligned}$$

An inductive limit preserves exactness (see Problem 5, below). Hence, we get

$$\begin{aligned}&0 \to \varinjlim_{x\in U} \mathcal{F}(U) \to \varinjlim_{x\in U} \mathcal{G}(U) \to \varinjlim_{x\in U} \mathcal{I}(U) \to 0,\\ &0 \to \varinjlim_{x\in U} \mathcal{I}(U) \to \varinjlim_{x\in U} \mathcal{H}(U) \to \varinjlim_{x\in U} \mathcal{C}(U) \to 0.\end{aligned}$$

Then Example 4.2 implies that the above exact sequences become

$$\begin{aligned}&0 \to (\operatorname{Ker}\varphi)_x \to \mathcal{G}_x \to (\operatorname{Im}\varphi)_x \to 0,\\ &0 \to (\operatorname{Im}\varphi)_x \to \mathcal{H}_x \to (\operatorname{Coker}\varphi)_x \to 0.\end{aligned}$$

Since φ_x is precisely the map $\mathcal{G}_x \to (\operatorname{Im}\varphi)_x \to \mathcal{H}_x$, which is obtained from the above exact sequences, the proof is completed. □

PROBLEM 5. Let Λ be a preordered set and let $\{L_\lambda, f_{\mu\lambda}\}$, $\{M_\lambda, g_{\mu\lambda}\}$ and $\{N_\lambda, h_{\mu\lambda}\}$ be inductive systems of additive groups indexed by Λ. For an exact sequence

$$0 \to L_\lambda \to M_\lambda \to N_\lambda \to 0, \quad \lambda \in \Lambda,$$

satisfying the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L_\lambda & \longrightarrow & M_\lambda & \longrightarrow & N_\lambda & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_{\mu\lambda}} & & \downarrow{\scriptstyle g_{\mu\lambda}} & & \downarrow{\scriptstyle h_{\mu\lambda}} & & \\ 0 & \longrightarrow & L_\mu & \longrightarrow & M_\mu & \longrightarrow & N_\mu & \longrightarrow & 0, \end{array}$$

prove that the sequence

$$0 \to \varinjlim_{\lambda\in\Lambda} L_\lambda \to \varinjlim_{\lambda\in\Lambda} M_\lambda \to \varinjlim_{\lambda\in\Lambda} N_\lambda \to 0$$

is exact.

Let $\mathcal{F}$ be a subsheaf of a sheaf $\mathcal{G}$. For the natural map $\iota : \mathcal{F} \to \mathcal{G}$, denote its cokernel by $\mathcal{G}/\mathcal{F}$. The sheaf $\mathcal{G}/\mathcal{F}$ is said to be the *quotient sheaf* of $\mathcal{G}$ by the subsheaf $\mathcal{F}$. The quotient sheaf $\mathcal{G}/\mathcal{F}$ is precisely the sheaf associated to the presheaf $\mathcal{G}(U)/\mathcal{F}(U)$ of additive groups.

EXAMPLE 4.6. Recall the subsheaf $\mathcal{J}$ of the structure sheaf $\mathcal{O}^1_{\mathbb{P}^k}$ of a one-dimensional projective space $\mathbb{P}^1_k$ over a field k (see Example 4.4). Since for $x \neq \mathfrak{p}_0, \mathfrak{p}_\infty$ we have $\mathcal{J}_x = \mathcal{O}_{\mathbb{P}^1_{k,x}}$, the associated sheaf $\mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J}$ of the presheaf $\mathcal{C}$ in Example 4.4 has a zero stalk (i.e., trivial additive group) at x. Furthermore, since $\mathcal{J}_{\mathfrak{p}_0}$ is the ideal of $\mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}_0}$ generated at x, we obtain $(\mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J})_{\mathfrak{p}_0} = \mathcal{O}_{\mathbb{P}^1_k,\mathfrak{p}_0}/\mathcal{J}_{\mathfrak{p}_0} \xrightarrow{\sim} k$. Therefore, the stalks of $\mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J}$ at $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$ are k, and 0 otherwise. This result gives an impression that skyscrapers stand only at $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$. Hence, the sheaf $\mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J}$ is sometimes called a *skyscraper sheaf*. We have $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J}) \xrightarrow{\sim} k \oplus k$. □

(c) **Exact Sequences**

Let

$$\cdots \to \mathcal{F}_{i-1} \xrightarrow{\varphi_{i-1}} \mathcal{F}_i \xrightarrow{\varphi_i} \mathcal{F}_{i+1} \xrightarrow{\varphi_{i+1}} \cdots \tag{4.9}$$

be a sequence of sheaves $\mathcal{F}_i$ of additive groups over a topological space X, where each φ_i is an additive group homomorphism. When $\operatorname{Im}\varphi_{i-1} = \operatorname{Ker}\varphi_i$ for each i, the sequence (4.9) is said to be *exact*. Let 0 be the sheaf which assigns a trivial additive group for each open set of X.

For a sheaf homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$, as in the case of additive groups, φ is said to be *injective* when $\operatorname{Ker}\varphi = 0$, and *surjective* when $\operatorname{Im}\varphi = \mathcal{G}$. In terms of exact sequences, $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if the sequence

$$0 \to \mathcal{F} \to \mathcal{G}$$

is exact and φ is surjective if the sequence

$$\mathcal{F} \to \mathcal{G} \to 0$$

is exact. Furthermore, the sequence

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0 \tag{4.10}$$

is exact when φ is injective, ψ is surjective, and $\operatorname{Im}\varphi = \operatorname{Ker}\psi$. When (4.10) is exact, it is called a *short exact sequence.* Short exact sequences will appear often in this book.

In particular, if $\mathcal{F}$ is a subsheaf of $\mathcal{G}$, the inclusion $\mathcal{F}(U) \subset \mathcal{G}(U)$ induces the injective natural map $\iota : \mathcal{F} \to \mathcal{G}$. Then we have the exact sequence

$$0 \to \mathcal{F} \xrightarrow{\iota} \mathcal{G} \to \mathcal{G}/\mathcal{F} \to 0.$$

This and the exact sequence (4.10) imply that $\mathcal{H}$ is isomorphic to $\mathcal{G}/\mathcal{F}$.

PROBLEM 6. Prove that an injective and surjective homomorphism $\varphi : \mathcal{F} \to \mathcal{G}$ is an isomorphism, i.e., for each open set U, $\varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ is an isomorphism.

From Theorem 4.5, we clearly obtain the following proposition.

PROPOSITION 4.7. *For sheaves of additive groups over a topological space X, the sequence (4.9) is exact if and only if the sequence of stalks at each x in X*

$$\cdots \to \mathcal{F}_{i-1,x} \xrightarrow{\varphi_{i-1,x}} \mathcal{F}_{i,x} \xrightarrow{\varphi_{i,x}} \mathcal{F}_{i+1,x} \xrightarrow{\varphi_{i+1,x}} \cdots$$

is an exact sequence of additive group homomorphisms. □

PROPOSITION 4.8. *For an exact sequence*

$$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H}$$

of sheaves of additive groups over a topological space X and for each open set U of X, we have the following exact sequence of additive groups:

$$0 \to \Gamma(U, \mathcal{F}) \xrightarrow{\varphi_U} \Gamma(U, \mathcal{G}) \xrightarrow{\psi_U} \Gamma(U, \mathcal{H}).$$

However, ψ_U need not be surjective even if ψ is surjective.

PROOF. For the induced sequence of additive groups

$$0 \to \Gamma(U, \mathcal{F}) = \mathcal{F}(U) \xrightarrow{\gamma_U} \Gamma(U, \mathcal{G}) = \mathcal{G}(U) \xrightarrow{\psi_U} \Gamma(U, \mathcal{H}) = \mathcal{H}(U),$$

since φ is injective, the sheaf property (F1) implies the injectivity of φ_U. Consider $t \in \mathcal{G}(U)$ such that $\psi_U(t) = 0$. Since $\operatorname{Ker}\psi_x = \operatorname{Im}\varphi_x$, there is $s_x \in \mathcal{F}_x$ satisfying $\varphi_x(s_x) = t_x$. Since φ_x is injective, this s_x is uniquely determined. Choose an open neighborhood V of x and $s_V \in \mathcal{F}(V)$ so that the germ of s_V at x is s_x. Then there exists a small enough open neighborhood $W \subset V$ such that $\rho_{W,V}(\varphi_V(s_V)) = \rho_{W,V}(t)$. Thus, we get an open covering $\{U_j\}_{j \in J}$ of U and $s_j \mathcal{F}(U_j)$ so that $\varphi_{U_j}(s_j) = \rho_{U_j,U}(t)$. Since φ_{U_j} is injective, s_j is uniquely determined. On $U_{jk} = U_j \cap U_k \neq \varnothing$, $\varphi_{U_{jk}}$ is also injective. Hence we have $\rho_{U_{jk},U_j}(s_j) = \rho_{U_{jk},U_k}(s_k)$. Therefore, there exists $s \in \mathcal{F}(U)$ such that $\phi_{U_j,U}(s) = s_j$, $j \in J$. Namely, $\operatorname{Im}\varphi_U = \operatorname{Ker}\psi_U$.

From Example 4.4, we have the exact sequence

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathbb{P}^1_k} \to \mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J} \to 0.$$

As shown in Example 4.4, we have $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) = k$. From Example 4.6, we have $\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J}) = k \oplus k$. Hence the homomorphism

$$\Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}) \to \Gamma(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}/\mathcal{J})$$

cannot be surjective. Therefore, for a surjective ψ, ψ_U need not be surjective. □

Hence, for a short exact sequence of sheaves

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0,$$

the sequence of sections over an open set U is guaranteed to be exact only as

$$0 \to \Gamma(U, \mathcal{F}) \to \Gamma(U, \mathcal{G}) \to \Gamma(U, \mathcal{H}).$$

This lack of exactness on the right necessitates the cohomology of sheaves, which makes algebraic geometry difficult and more interesting.

Note that, as explained above, for a surjective sheaf homomorphism $\psi : \mathcal{G} \to \mathcal{H}$ and for a section $t \in \mathcal{H}(U)$ over an open set U, at each point x in U there exists an open neighborhood $V \subset U$ of x satisfying $\psi_V(s_V) = \rho_{V,U}(t)$ for some $s_V \in \mathcal{G}(V)$. In general $V \neq U$, i.e., ψ_U need not be surjective. One can find an open covering $\{U_j\}_{j \in J}$ of U and $s_j \in \mathcal{G}(U_j)$ so that $\psi_{U_j}(s_j) = \rho_{U_j,U}(t)$. Cohomology's role is to describe when one can choose $\{s_j\}$ to get $s \in \mathcal{G}(U)$ and $\psi_U(s) = t$.

EXAMPLE 4.9. Let $\mathcal{O}_X$ be the sheaf of holomorphic functions over the complex plane $X = \mathbb{C}$ (i.e., regular functions). See Example 2.18(3). Let $\mathcal{M}_X$ be the sheaf of meromorphic functions associated to the set of all meromorphic functions on an open set U of X (locally the quotient f/g of holomorphic, i.e., regular functions). By the natural inclusion $\mathcal{O}_X(U) \subset \mathcal{M}_X$, $\mathcal{O}_X$ may be considered as a subsheaf of $\mathcal{M}_X$. We have the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{M}_X \to \mathcal{M}_X/\mathcal{O}_X \to 0. \tag{4.11}$$

For an open set U, we will study an element of $\Gamma(U, \mathcal{M}_X/\mathcal{O}_X)$. By the sheafification (4.1) of a presheaf, $t \in \Gamma(U, \mathcal{M}_X/\mathcal{O}_X)$ may be considered such that the $t_j \in \mathcal{M}_X(U_j)/\mathcal{O}(U_j)$, $j \in J$, where $\{U_j\}_{j \in J}$ is a properly chosen open covering of U, satisfy the following. Namely, the restrictions of t_j and t_k on $U_{jk} = U_j \cap U_k \neq \varnothing$ coincide. Let $\widetilde{t_j}$ be a meromorphic function on U_j satisfying $t_j \equiv \widetilde{t_j} \bmod \mathcal{O}_X(U_j)$. Then we get $\widetilde{t_j} - \widetilde{t_k} \in \mathcal{O}(U_{jk})$. On the other hand, a meromorphic function on U_j can have only isolated poles. If necessary, take a smaller U_j so that in U_j, $\widetilde{t_j}$ may have finitely many poles $a_1^{(j)}, \ldots, a_{n_j}^{(j)}$ whose principle part (the negative exponent terms) of the Laurent expansion is

$$p_i^{(j)} = \frac{\alpha^{(j)}_{k_j^{(i)}}}{(z - a_i^{(j)})^{k_j^{(i)}}} + \frac{\alpha^{(j)}_{k_j^{(i)}-1}}{(z - a_i^{(j)})^{k_j^{(i)}-1}} + \cdots + \frac{\alpha^{(j)}_{-1}}{z - a_i^{(j)}}.$$

Then $\widetilde{t_j} - \sum_{i=1}^{n_j} p_i^{(j)}$ is holomorphic in U_j. Therefore,

$$t_j \in \mathcal{M}_X(U_j)/\mathcal{O}_X(U_j),$$

which is determined by $\widetilde{t_j}$, associates points $a_i^{(j)}$, $1 \le i \le n_j$, and the principal part $p_i^{(j)}$ of the Laurent expansion. On $U_{jk} \neq \varnothing$, $\widetilde{t_j} - \widetilde{t_k}$ being holomorphic means that the principal parts of the Laurent expansions of $\widetilde{t_j}$ and $\widetilde{t_k}$ at the poles in U_{jk} coincide. Hence, to give $t \in \Gamma(U, \mathcal{M}_X/\mathcal{O}_X)$ is to give a sequence $\{a_\lambda\}$, without an accumulating point in U, and the principal part

$$\frac{\alpha^{(\lambda)}_{k_\lambda}}{(z - a_\lambda)^{k_\lambda}} + \frac{\alpha^{(\lambda)}_{k_\lambda - 1}}{(z - a_\lambda)^{k_\lambda - 1}} + \cdots + \frac{\alpha^{(\lambda)}_{-1}}{z - a_\lambda} \tag{4.12}$$

of the pole at a_λ. In the induced exact sequence

$$0 \to \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{M}_X) \xrightarrow{f} \Gamma(U, \mathcal{M}_X/\mathcal{O}_X)$$

from the exact sequence (4.11), in order for $t \in \Gamma(U, \mathcal{M}_X/\mathcal{O}_X)$ to be the image of the above homomorphism f, t must have (4.12) as the principal part of the Laurent expansion at a_j. There also exists a meromorphic function that is holomorphic in $U \setminus \{a_\lambda\}$. The Mittag-Leffler theorem in complex analysis implies that f is indeed surjective. □

We can extend the above example to the case of a domain D in $\mathbb{C}^n$. For $n \geq 2$, the poles of a meromorphic function are not isolated. Hence, the situation is more complicated. Then an element of $\Gamma(D, \mathcal{M}_D/\mathcal{O}_D)$ is said to be a *Cousin distribution*. The Cousin problem is to determine whether the Cousin distribution is the image of a meromorphic function or not. A Cousin problem is one of the intriguing problems for the development of the theory of holomorphic functions in several complex variables.

The above sheaf $\mathcal{M}_X$ corresponds to the *sheaf field of fractions* $\mathcal{K}_X$ for a scheme X.

In general, for a commutative ring R, the totality S of non-zero-divisors is multiplicatively closed. Then $S^{-1}R$ is said to be the *ring of total quotients*, denoted by $Q(R)$. If R is an integral domain, then $S = R \setminus \{0\}$. Then the ring of total quotients is exactly the *quotient field*. When R possesses a zero divisor, the ring of total quotients $Q(R)$ is not a field.

For an affine open set U of a scheme X, the ring of total quotients $Q(\Gamma(U, \mathcal{O}_X))$ of $\Gamma(U, \mathcal{O}_X)$ defines a presheaf. Let $\mathcal{K}_X$ be the associated sheaf, i.e., the sheafification of the presheaf $Q(\Gamma(U, , \mathcal{O}_X))$. Note also that for affine open sets $V \subset U$, the restriction map $\Gamma(U, \mathcal{O}_X) \to \Gamma(V, \mathcal{O}_X)$ induces a homomorphism $Q(\Gamma(U, \mathcal{O}_X)) \to Q(\Gamma(V, \mathcal{O}_X))$ of rings of total quotients.

EXERCISE 4.10. For an affine open set U of a Noetherian scheme X, we have

$$\Gamma(U, \mathcal{K}_X) = Q(\Gamma(U, \mathcal{O}_X)).$$

Furthermore, for each $x \in X$, we also have $\mathcal{K}_{X,x} = Q(\mathcal{O}_{X,x})$.

PROOF. Let $U = \operatorname{Spec} R$ be an affine open set of X, where R is a Noetherian ring. Choose $f_1, f_2, \ldots, f_n \in R$ so that $\{U_i = D(f_i)\}$, $i = 1, 2, \ldots, n$, is an open covering of U. Namely, $1 \in (f_1, f_2, \ldots, f_n)$.

We will prove the following two assertions.

(1) If the image in $Q(R_{f_i})$ of $\alpha \in Q(R)$ is 0 for $1 \leq i \leq n$, then $\alpha = 0$.

(2) For arbitrary i and j, if the image in $Q(R_{f_i f_j})$ of $\alpha_i \in Q(R_{f_i})$ coincides with the image in $Q(R_{f_i f_j})$ of $\alpha_j \in Q(R_{f_j})$, then there exists $\alpha \in Q(R)$ whose image in $Q(R_{f_i})$ is α_i.

The above assertions imply that for an affine open set U of X, the presheaf $Q(\Gamma(U, \mathcal{O}_X))$ is a sheaf, i.e., it satisfies (F1) and (F2). Therefore, for an affine open set, we have $\Gamma(U, \mathcal{K}_X) = Q(\Gamma(U, \mathcal{O}_X))$.

PROOF OF (1). Let $\alpha = \frac{b}{a}$, $a, b \in R$, where a not a zero divisor. From the hypothesis, there exists a positive integer m_i so that $f_i^{m_i} b = 0$. Let $m = \max_{1 \le i \le n} m_i$. Then, for all i, we get $f_i^m b = 0$. Since $1 \in (f_1, \dots, f_n)$, we have $1 = \sum_{i=1}^n a_i f_i$, $a_i \in R$. The nm-th power of both sides gives $1 = \sum_{i=1}^n c_i f_i^m$ for some $c_i \in R$. Then, $b = 1 \cdot b = \sum c_i (f_i^m b) = 0$, i.e., $\alpha = 0$. □

PROOF OF (2). Let $\alpha_i = \frac{b_i}{a_i}$, $a_i, b_i \in R_{f_i}$. If necessary, by replacing a_i and b_i by $f_i^l a_i$ and $f_i^l b_i$, respectively, one may assume $a_i, b_i \in R$. When $\frac{b_i}{a_i} = \frac{b_j}{a_j}$ over $U_i \cap U_j = D(f_i) \cap D(f_j) = D(f_i f_j)$, we can find N satisfying $(f_i f_j)^N (a_i b_j - a_j b_i) = 0$. Then, by multiplying a power of f_i to a_i and b_i, we can assume that $a_i b_j - a_j b_i = 0$ for all i and j. Let

$$I = \{ r \in R \mid \text{ for all } i,\ r b_i \text{ belongs to the ideal } (a_i) \text{ of } R_{f_i} \}.$$

Then I is an ideal. Since $a_j b_i = a_i b_j \in (a_i)$, we have $a_1, a_2, \dots, a_n \in I$. The Noetherianness of R implies that we can write $I = (c_1, \dots, c_s)$. If $cc_j = 0$, $1 \le j \le n$, then since $a_i \in I$, we get $ca_i = 0$, $1 \le i \le s$. Since a_i is not a zero divisor in R_{f_i}, c is 0 in R_{f_i}. Namely, there exists a positive integer M so that $f_i^M c = 0$ for all i. As before, we conclude that $c = 0$. Thus I contains a non-zero-divisor α. The definition of I implies that there is $\alpha_i \in R_{f_i}$ to satisfy $\alpha b_i = \alpha_i a_i$. That is, $\frac{\alpha b_i}{a_i} \in \Gamma(U_i, \mathcal{O}_X)$. Since $\frac{b_i}{a_i} = \frac{b_j}{a_j}$ over $U_i \cap U_j$, we get $\frac{\alpha b_i}{a_i} = \frac{\alpha b_j}{a_j}$. Then $\frac{\alpha b_i}{a_i}$, $1 \le i \le n$, define an element β in $\Gamma(U, \mathcal{O}_X)$. Therefore, the image of $\frac{\beta}{\alpha} \in Q(\Gamma(U, \mathcal{O}_X))$ in $\Gamma(U_i, \mathcal{O}_X)$ is $\frac{b_i}{a_i}$, i.e., $\frac{\beta}{\alpha}$ is the element that we seek. The last claim is obvious from $\Gamma(U, \mathcal{K}_X) = Q(\Gamma(U, \mathcal{O}_X))$. □

Since for an affine open set U, $Q(\Gamma(U, \mathcal{O}_X))$ is a $\Gamma(U, \mathcal{O}_X)$-module, $\mathcal{K}_X$ is an $\mathcal{O}_X$-module. However, $\mathcal{K}_X$ is not a quasicoherent $\mathcal{O}_X$-module in general (see the next section for quasicoherent sheaves).

4.2. Quasicoherent Sheaves and Coherent Sheaves

All sheaves that we have seen so far are sheaves of additive groups. In this section, we will study sheaves of $\mathcal{O}_X$-modules. Then we will focus on quasicoherent sheaves and coherent sheaves, which play an important role in algebraic geometry. Even though the theory can be built on ringed spaces, we will develop it over schemes.