Finally, to show that the series (6) is a *bona fide* generalization of Taylor's series, note that for $\phi_i(x) = 1$, for which Conditions B are satisfied, (6) reduces to Taylor's series. Conditions B are also satisfied if

$$\phi_i(x) = e^{-\frac{x}{n}},$$

and for these values one is led to a series quite different from Taylor's series.

* NATIONAL RESEARCH FELLOW.

[1] *Bul. Sci. Math. Astr.*, **25,** p. 200.

[2] E. Goursat, *Cours d'Analyse Mathématique,* vol. 2, p. 430.

[3] *Math. Ann.*, **75,** p. 449.

A THEORY OF MATTER AND ELECTRICITY[1]

BY GEORGE D. BIRKHOFF

DEPARTMENT OF MATHEMATICS, HARVARD UNIVERSITY

Communicated January 22, 1927

Up to the present time no mathematical theory of matter and electricity seems to have been proposed which meets the fundamental demands of *determinateness* and *stability*. A theory which appears to satisfy these demands is presented herewith. In a second following note it is proved that this theory leads to a formula of the Balmer type for the frequencies of the small oscillations of a hydrogen atom.

1. *The Perfect Fluid.*—We consider first the space-time of the special theory of relativity with time coördinate x_1 and rectangular space coördinates x_2, x_3, x_4, the units of length and time being so chosen that the velocity of light is 1. By an "adiabatic fluid" is meant matter whose state is determined by a functionally related pressure and density, p and ρ such that if $u^i = dx_i/ds$ denotes the velocity tensor, and if the "energy tensor" is defined to be

$$T^{ij} = \rho u^i u^j - pg^{ij} \quad (g^{ij} = 0, i \neq j;\ g^{11} = -g^{22} = -g^{33} = -g^{44} = 1), \tag{1}$$

then the equations of motion are obtained by setting

$$\frac{\partial T^{i\alpha}}{\partial x_\alpha} = 0. \tag{2}$$

These equations express the relativistic form of the principles of conservation of energy ($i = 1$) and of linear momentum in the direction of the three axes ($i = 2, 3, 4$). From them it appears that a certain quantity

$$\int \rho e^{-\int \frac{dp}{\rho}} d\tau \quad (d\tau, \text{ element of volume}) \tag{3}$$

has always the same value over any given portion of the fluid;[2] for example, if $p = \rho$, it is the volume $\int d\tau$ which is invariable. Under the action of an arbitrary body force tensor f^i, the only modification necessary is to replace the right-hand member of (2) by f^i. It is understood that the only forces allowed must be orthogonal to the velocity tensor, i.e., $f^\alpha u_\alpha = 0$.

A possible state of equilibrium of such a fluid is that of constant pressure and density. If one computes the velocity of a slight disturbance from this state, it turns out to be $a^{1/2}/(1-a)^{1/2}$ where we write $a = dp/d\rho$.[3] Of course, the velocity of such an elastic disturbance cannot exceed that of light. On the other hand, if that velocity is less than that of light, difficulties of indeterminateness seem to arise when two portions of the adiabatic fluid collide at relative velocities sufficiently near the velocity of light. In order to avoid this difficulty of indeterminateness, it seems to be necessary to have a disturbance velocity equal to that of light at all pressures and densities. Such a fluid, for which $p = {}^1/_2\rho$, will be termed the "perfect fluid," by analogy with an ordinary perfect gas.

2. *Electricity and the Perfect Fluid.*—If F_{ij} is a skew-symmetric tensor of the second order so that $F_{ij} = -F_{ji}$ then the tensor

$$f^i = F^{i\alpha}u_\alpha$$

yields a force tensor which is in all cases orthogonal to the velocity tensor. We will regard F_{12}, F_{13}, F_{14} as the components of electric force and $-F_{34}$ $-F_{42}$, $-F_{23}$ as the components of magnetic force. The well-known Maxwell-Lorentz equations may then be written in the form

$$\frac{\partial F_{ij}}{\partial x_k} + \frac{\partial F_{jk}}{\partial x_i} + \frac{\partial F_{ki}}{\partial x_j} = 0, \quad \frac{\partial F_i^{\ \alpha}}{\partial x_\alpha} = -4\pi\sigma u_i, \tag{4}$$

where σ is the density of electricity.

From these it follows that the total charge σ is invariable, so that electricity may be regarded as a substance.

Now we suppose that the perfect fluid is permanently charged with electricity of which it acts as the carrier. More precisely if we define the "electromagnetic energy tensor" as

$$E_{ij} = F_i^{\ \alpha}F_{\alpha j} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g_{ij}, \tag{5}$$

the ponderomotive equations for combined matter and electricity take a form like (2) except that the energy tensor T_{ij} of matter is replaced by $T_{ij} + E_{ij}$, i.e., by the combined energy tensor of the perfect fluid and of electricity.

Since the quantity (3), as well as the electrical charge is invariable it follows that these must remain in a constant ratio along any world line of the fluid. For the case of the perfect fluid this leads at once to the equation (6)

$$\rho = \frac{\varphi^2 \sigma^2}{\pi}. \tag{6}$$

Here $\varphi/\sqrt{\pi}$ is the constant ratio referred to with the divisor $\sqrt{\pi}$ introduced for the sake of convenience. The quantity φ will be termed the "substance coefficient."[4] It is an arbitrary function of position, specified once for all at the outset and remaining fixed in value along each world line of the fluid.

Since electricity of one sign repels itself, and since the perfect fluid tends to expand under the enormous elastic pressures, the charged perfect fluid tends to expand all the more with velocities which approach that of light. Under such expansion the mass of the relativistic fluid approaches zero since the density of matter varies as the square of the density of electricity by (6).

3. *Atomic Potential Energy.*—As a first step toward relieving the instability just referred to we propose to define an "atomic potential" ψ, given once for all at each particle of the fluid. In this case along the world line of a particle we have

$$\frac{d\psi}{ds} = \frac{\partial \psi}{\partial x_\alpha} u^\alpha = 0$$

so that the tensor $\partial\psi/\partial x_i$ may function as a second force tensor, being orthogonal to the velocity tensor. It is this further body force which we shall assume to be present in the fluid. If we define

$$A_{ij} = \psi g_{ij} \tag{7}$$

as the "atomic potential energy tensor," then the laws of motion for the perfect fluid under elastic, electrical and atomic potential forces may be expressed in the form (2) except that T_{ij} is replaced by the complete energy tensor

$$H_{ij} = T_{ij} + E_{ij} + A_{ij},$$

i.e., the equations of motion are

$$\frac{\partial H^{i\alpha}}{\partial x_\alpha} = 0. \tag{8}$$

It is to be remembered that in addition to these equations there is the constitutive equation $p = {}^1/_2\rho$ as well as the Maxwell-Lorentz equations (4), and the equation

$$\frac{\partial \psi}{\partial x_\alpha} u^\alpha = 0, \tag{9}$$

expressing the fact that ψ is constant along each world line.

It would seem to be desirable that ψ be required to vanish along the free boundary of any portion of the perfect fluid. Otherwise there will be an inward normal pressure equal to ψ, and the resultant of the internal body forces due to the atomic potential will not vanish.

4. *The Conservation of Energy.*—If one transforms properly the equation (8) for $i = 1$, it leads to a principle of the conservation of energy at low velocities in which

$$V = \int \left[\rho - p + \frac{1}{8\pi}(F_{12}^2 + \ldots + F_{23}^2) + \psi \right] d\tau \qquad (10)$$

figures as potential energy, so that only half the mass counts an elastic energy. The electromagnetic energy density has the usual form.

The atomic potential energy is proportional to the volume, and in consequence the volume cannot increase indefinitely. Here then is a first factor operating in the direction of stability. A portion of the fluid of one sign would tend to scatter in wisplike form in space rather than to fill all of space.

For $i = 2, 3, 4$ the equation (8) expresses the principles of the conservation of linear momentum in the direction of the x_2, x_3, x_4 axes, respectively.

5. *Gravitation.*—If now, in accordance with Einstein's general scheme for taking care of gravitational phenomena, we write

$$R_{ij} - \frac{1}{2} R g_{ij} = - 8\pi H_{ij}, \qquad (11)$$

and employ the same laws as before in the sense of the "principle of equivalence," an appropriate extension of the above theory of matter and electricity to the space-time of the general theory of relativity results. As is well known, the relations (8) then appear as implicit in (11). The gravitational effect of matter and energy is, however, not exactly proportional to the energy density (see (10)) but is instead given by

$$\rho + \frac{1}{8\pi}(F_{12}^2 + \ldots + F_{23}^2) + \psi. \qquad (12)$$

6. *The Structure of Matter. Stability.*—To obtain a theory of matter satisfying the fundamental requirements of determinateness and stability we may now proceed as follows.

Let us suppose that the protons are initially portions of the perfect fluid carrying a charge e, and of one or more types of spherical distribution of φ, ψ, σ. These protons may collide with one another or with themselves, but by definition are not to interpenetrate.

Similarly, certain other portions of the perfect fluid of one particular type carrying a charge $-e$ constitute the electrons. Their mass is much less and the volume much greater than that of the protons. These may collide with one another, but by definition they cannot interpenetrate.

If now we were to require that electrons and protons cannot interpenetrate it is readily proved that they tend to wisplike forms of finite volume with nearly neutralizing portions of positive and negative electricity in every part. The natural way in which to avoid this difficulty of an amorphous stable condition is to allow the free interpenetration of electron and proton. It is almost as though the proton were taken as a point charge which freely penetrates the electron, although this limiting case introduces the vital difficulty of infinite available energy.

We, therefore, require the free interpenetration of the proton and electron. Howevei, there can be no three fluids overlapping at a point, since two of the same sign cannot overlap. The appropriate modification in the equations of motion is easily made. In the Maxwell-Lorentz equations (4) the current density tensor σu_i is replaced by $\sigma_+ u_i + \sigma_- v_i$ where σ_+ is the density of positive charge with velocity tensor u_i while σ_- is the density of the negative charge with velocity tensor v_i. Similarly the energy tensors T_{ij} and A_{ij} are modified, respectively, to

$$T_{ij}^* = \rho_+ u_i u_j + \rho_- v_i v_j - (p_+ + p_-)g_{ij}, \; A_{ij}^* = (\psi_+ + \psi_-)g_{ij}$$

where the meaning of the notation is obvious.

The equations (8) express as before the four conservation conditions, and follow at once from the equations of motion for the interpenetrating fluids. For example, for the proton these are

$$\frac{\partial T^{i\alpha}}{\partial x_\alpha} = f^i$$

where f^i is the electromagnetic force tensor for the proton,

$$f_i = \sigma_+ F_{i\alpha} u^\alpha.$$

In this way, even in the space-time of the general theory of relativity, the modified equations are at once obtained.

With such a set of requirements it is not hard to see that a position of stable equilibrium of any system of protons and electrons can be expected to appear. In fact none of the protons or electrons can contract indefinitely since that would require an infinite elastic potential energy. Nor can they expand indefinitely because of the atomic potential energy. And, finally, the electrical attractions between the oppositely charged protons and electrons, and their free interpenetration, tend to make them coalesce into neutral stable forms made up of a numbei of connected pairs of protons and electrons forming the individual atoms.

[1] An outline of this theory was presented in an address entitled "A Mathematical Critique of Some Physical Theories," given before a meeting of Section A of the American Association for the Advancement of Science, The American Mathematical Society and the Mathematical Association of America on Dec. 30, 1926. The address will appear in full in an early number of the *Bulletin of the American Mathematical Society*.