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Arrangements, Local Systems and Singularities

CIMPA Summer School, Galatasaray University, Istanbul, 2007

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Preface

This volume comprises a set of lecture notes from the CIMPA Summer School, *Arrangements and Local systems and Singularities*, held at Galatasaray University, Istanbul, during June 11-22, 2007. The school was attended by 68 mathematicians, 35 of them from 19 countries outside Turkey. The Summer School was made up of eleven short courses and five seminars presented by an outstanding group of lecturers who covered a wide range of topics related to the concepts of arrangements, local systems and singularities. The list of lectures of the workshop appears below. Most members of the audience were graduate students or young researchers and the primary purpose of the school was to introduce them, in particular those from developing countries, to the many advances and opportunities in this widely applicable field.

Historically, research on arrangements of hyperplanes, starting with the work of V.I. A'rnold and E. Brieskorn and later Peter Orlik, was presented at meetings devoted mainly to discussions of singularities. Some of the articles in this volume demonstrate that the interaction between these two subjects is ongoing and particularly productive. Beyond this, the volume is intended for a large audience in pure mathematics, including researchers and graduate students working in algebraic geometry, singularity theory, topology and related fields. The reader will find in the Problem Section at the end of the volume a variety of open problems proposed by the lecturers at the end of the School's sessions as directions ripe for further study and development.

We would like to thank the *Centre des Mathématiques Pures et Appliquées (CIMPA)* for sponsoring the School and Professor Richard Grin for organizational help. We would also like to thank the *Scientific and Technological Research Council of Turkey (TUBITAK)* for their financial help which made it possible to invite 16 young mathematicians from Turkey as well as some lecturers from abroad. We were able to support participants from across the region, thanks to the generous financial help provided by the *International Center for Theoretical Physics (ICTP)* and the *International Mathematical Union (IMU).*

During the long preparatory process and also during the school, Gülay Kaya, Ayşe Altıntaş and Celal Cem Sarıoğlu contributed at various levels to the organization. We are grateful to them.

F. El Zein, A. Suciu, M. Tosun, A.M. Uludağ and S. Yuzvinsky

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List of Lectures of the School

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D. Cohen
Fundamental groups of complements of hyperplane arrangements.

G. Denham
Homological algebra of hyperplane arrangements.

F. El Zein
Local systems, constructible sheaves and Geometry.

M. Falk *Geometry and combinatorics of resonance weights.*

G. Kaya
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A.U.O. Kisisel
Toric varieties and the diagonal properties.

Lê D.T.
Simple singularities and simple Lie algebras.

D. Matei
Artin groups, Bestvina-Brady groups and arrangements of hypersurfaces.

D. Mond
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(2) *Conservation of multiplicity. Cohen-Macaulay rings. Free divisors (from groups).*

M. Oka
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(2) *Periodicity of arrangements with integral coefficients modulo positive integers.*

M. Yoshida
Hyperbolic Schwarz map for the hypergeometric differential equation.

M. Yoshinaga
Minimal CW decomposition of the complement of hyperplane arrangements.

S. Yuzvinsky
Orlik-Solomon algebras of hyperplane arrangements.

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Combinatorics of Covers of Complexified Hyperplane Arrangements

Emanuele Delucchi

Abstract. This is a survey of combinatorial models for covering spaces of the complement of a complexified hyperplane arrangement. We obtain a unified picture of the subject, and a generalization of various known results, by exploiting the toolkit of homotopy colimits for combinatorial applications developed by Welker, Ziegler and Živaljević.

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Keywords. Arrangements of hyperplanes, oriented matroids, order complexes of posets, Salvetti complex, homotopy colimits, combinatorial topology.

Introduction

A *cover of an arrangement* is a topological cover of the space obtained by removing a finite set of hyperplanes from a complex, finite-dimensional vector space.

The study of combinatorially defined complexes modeling covers of arrangements has a story that goes back to the beginnings of the topological theory of hyperplane arrangements, and arises in the context of finite real reflection groups, where one can consider the set of hyperplanes ('mirrors') fixed by the reflections in the group. In 1971 Brieskorn [16] conjectured the complement of the complexification of this set of hyperplanes to be an aspherical space (we then say that this is a $K(\pi, 1)$-*arrangement*).

Brieskorn's conjecture was settled by a general theorem of Deligne [29], who proved that the complexification of any real arrangement of linear hyperplanes whose chambers are simplicial cones is $K(\pi, 1)$. The idea was to prove contractibility of the universal covering space of the arrangement's complement, and the method involved designing a cell complex that, under certain conditions, models the universal cover of the arrangement's complement.

The $K(\pi, 1)$ problem for hyperplane arrangements, i.e., the problem of deciding whether the property of being $K(\pi, 1)$ is determined by the combinatorics of the lattice of intersections of the hyperplanes, is still open and in the focus of active research. The construction and the study of different models for the universal covering space of arrangement complements has been one of the main strategies used to attack this problem. Alternative approaches have been successfully exploited, most notably the idea to reduce the problem to a lower dimensional situation by linear fibrations (that led to the concept of supersolvable arrangements [76, 37]), the use of fibrations onto the complex torus [54], or a mix of the different techniques [21]. For a general reading on the $K(\pi, 1)$ problem for arrangements we point to the survey of Falk and Randell [38, 39].

Among more recent topics in arrangement theory are the study of local system cohomology on arrangement complements and of the topology of the Milnor fiber. In both these subjects, the homology of certain covering spaces plays an important role (see e.g. [27] and [30]).

For general complex arrangements not much is known. Björner and Ziegler [11] described a simplicial model for the complement of a complex arrangement, but no description of the covering space is at hand. After previous partial results of Nakamura [53], the case of finite complex reflection arrangements was recently settled by Bessis [3], who described a model for the universal cover of the orbit space and showed its contractibility using the theory of Garside groups and Garside categories, thus proving the $K(\pi, 1)$ conjecture for this class of arrangements (for more details see Remark 5.15 and Section 6).

In this survey we present a unified view on the different combinatorial models for covers of complexified real arrangements.

We put the subject into the framework of the theory of diagrams of spaces and homotopy colimits for combinatorial applications, as developed by Welker, Ziegler and Živaljević in [81]. Diagrams of spaces have already been fruitfully exploited to study the *link* of hyperplane arrangements (i.e., the space defined by the union of the hyperplanes) [82, 77, 46]. In our context, these techniques allow us for instance to link the two main classes of complexes we will be dealing with, namely the Salvetti-type models W_ρ (Definition 3.1) and the Garside-type models U_ρ (Definition 5.1). Each of these types of models generalizes some known constructions, that we will explain. We thus obtain a unified picture of the subject. Moreover, this language allows us to apply the known techniques for the study of the homotopy type of diagram of spaces.

We will use some facts from the covering theory of groupoids. Also, we will meet along our way the notion of oriented systems (with a corresponding covering theory) as introduced by Paris [59]. We hope that the chosen notation and the explanations will succeed in clarifying the interplay among the different notions of "cover", nevertheless avoiding confusion.

We will begin our exposition by recalling some definitions and facts that are nowadays standard in arrangement theory. In Section 2 we introduce diagrams of spaces and their homotopy colimits and state some basic facts about them.

Then, in Section 3 we will present a first type of diagram models and study their homotopy colimits. For every topological cover of the given arrangement we construct a diagram of which homotopy colimit is isomorphic to the given covering space, and can be written as the order complex of an explicitly described poset. These models are called of *Salvetti type* because the model of the identical cover is actually isomorphic to the complex introduced by Salvetti in [67]. Specializing to the universal covering of arrangements of linear hyperplanes we recover naturally the simplicial complex obtained by Luis Paris in [60]. Moreover, we will mention here the work of Charney and Davis on Artin groups [22, 23], also pointing to an application of it given by Charney and Peifer [24] in the context of affine reflection arrangements.

In fact, Paris constructed topological models for arbitrary covers of linear arrangements [59]. In Section 4 we first explain this construction. Then, we describe a stratification of it whose nerve is isomorphic to the poset obtained from the diagram model of the corresponding cover, thereby showing that the diagram models offer a compact and handy description (in fact, as order complexes of posets) of Paris' models.

Restricting our attention from affine to linear real arrangements, Section 5 introduces another type of diagram models generalizing a construction that arose in the context of Garside groups [14, 5, 25]. We call them therefore of *Garside type*. As an application, we then explain how Deligne's argument can be reformulated in view of this type of models. The closing section is about possible further applications and directions of work.

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1. Definitions and background

1.1. Arrangements

We will denote by $\mathcal{A}$ a collection of affine hyperplanes in $\mathbb{R}^d$, also called a *real arrangement*. Our considerations will be restricted to the case where the arrangement is locally finite (i.e., every point of $\mathbb{R}^d$ is contained in at most finitely many hyperplanes) and essential (i.e., the minimal intersections of hyperplanes are points).

The classical reference on arrangements of hyperplanes is the textbook of Orlik and Terao [56], and for the combinatorics of real arrangements in terms of oriented matroids we point to [10]. Let us here only recall the facts that we will need.

The closed strata that are determined by $\mathcal{A}$ in $\mathbb{R}^d$ are the *faces* of $\mathcal{A}$. The *support* of a face F is the set $\mathrm{supp}(F)$ of all hyperplanes containing F. The set of faces of $\mathcal{A}$ is partially ordered by reverse inclusion, so that for any two faces F_1, F_2 we have $F_1 \geq F_2$ if and only if $F_1 \subseteq F_2$: this defines the poset of faces of the arrangement, denoted by $\mathcal{F}(\mathcal{A})$.

The minimal elements of $\mathcal{F}(\mathcal{A})$ are the connected components that are cut out in $\mathbb{R}^d$ by $\mathcal{A}$ and are usually called *chambers*, or *regions*. Given two chambers C_1, C_2 of $\mathcal{A}$, one may choose a point in the interior of each chamber and consider the line segment spanned by these points. The hyperplanes that are met by this segment *separate* C_1 from C_2; the set of all hyperplanes separating C_1 from C_2 is denoted by $S(C_1, C_2)$. Two chambers are said to be *adjacent* if they are separated by only one hyperplane. If the arrangement is linear, we define the *opposite* of a chamber C to be the unique chamber $-C$ so that $S(C, -C) = \mathcal{A}$. If C is any chamber and F any face of $\mathcal{A}$, we will denote by C_F the unique chamber that contains F in its closure, and that is not separated from C by any of the hyperplanes that contain F – i.e, such that $C_F < F$ and $S(C_F, C) \cap \mathrm{supp}(F) = \emptyset$. The set of all regions of $\mathcal{A}$ will be written $\mathcal{T}(\mathcal{A})$ and can be given different partial orderings, depending on the choice of a base element. Once a *base chamber* $B \in \mathcal{T}$ is fixed, an associated partial order $\prec_B$ can be defined by setting $C_1 \preccurlyeq_B C_2$ if and only if $S(B, C_1) \subseteq S(B, C_2)$. This gives rise to the *poset of regions of $\mathcal{A}$ with base B* (introduced in [35]), that we will denote by $\mathcal{T}_B(\mathcal{A})$.

The *arrangement graph* $G(\mathcal{A})$ has $\mathcal{T}(\mathcal{A})$ as its set of vertices, and it is constructed by putting two opposite oriented edges between each pair of vertices that represent adjacent chambers. As an example, see the left side of Figure 1 for a picture of $G(\mathcal{A})$ when $\mathcal{A}$ consists of two lines in the plane. A directed path in the arrangement graph is called *positive*; it is called also *minimal* if it does not "cross" twice any hyperplane.

The *complexification* of the arrangement $\mathcal{A}$ is the set $\mathcal{A}_{\mathbb{C}}$ of the complex hyperplanes obtained by considering the same (real) defining equations as for the elements of $\mathcal{A}$. We will be interested in the topology of the complement of the complexification (sometimes just called *the arrangement's complement*)

$$\mathcal{M}(\mathcal{A}) := \mathbb{C}^d \setminus \bigcup \mathcal{A}_{\mathbb{C}}.$$

1.2. Posets

We give a short review of some basic facts and notations about partially ordered sets (or, for short, *posets*). For a careful exposition of the subject see [75]. Given two elements x, y in a poset $\mathcal{P}$ we denote by $x \vee y$ their unique least upper bound (or *join*) and by $x \wedge y$ their unique maximal lower bound (or *meet*), if these exist. A poset where the meet and the join exist for every pair of elements is called a *lattice*. Given two posets $\mathcal{P}$ and $\mathcal{Q}$, we will partially order their disjoint union $\mathcal{P} \coprod \mathcal{Q}$ by letting $x \geq y$ if and only if either both $x, y \in \mathcal{P}$ and $x \leq y$ in $\mathcal{P}$, or $x, y \in \mathcal{Q}$ and $x \leq y$ in $\mathcal{Q}$. The main topological object associated to a poset $\mathcal{P}$ is its *order complex* $\Delta(\mathcal{P})$, that is the simplicial complex of the totally ordered subsets of $\mathcal{P}$. It is clear that if $\mathcal{P}$ has a unique minimal element $\hat{0}$, then the order complex $\Delta(\mathcal{P})$ will be a cone with apex $\hat{0}$, and thus in particular contractible. The analogous statement holds of course when $\mathcal{P}$ has a unique maximal element.

1.3. The Salvetti complex

We introduce the tool that will allow us to link the topology of $\mathcal{M}(\mathcal{A})$ to the combinatorics of the real arrangement. Let us begin with the abstract definition.

Definition 1.1. Let $\mathcal{S}(\mathcal{A})$ be the set of all pairs (F, C) with $F \in \mathcal{F}$, $C \in \mathcal{T}$ and $C < F$. We give this set a partial order by setting $(F_1, C_1) > (F_2, C_2)$ if and only if $F_1 > F_2$ in $\mathcal{F}$ and $C_2 = (C_1)_F$. The (simplicial version of the) *Salvetti complex* is

$$\mathrm{Sal}(\mathcal{A}) := \Delta(\mathcal{S}(\mathcal{A})).$$

The importance of this object lies in the following fundamental theorem, that was proved by Mario Salvetti.

Theorem 1.2 (Theorem 1 of [67]**).** *For every real arrangement $\mathcal{A}$, the geometrical realization of* $\mathrm{Sal}(\mathcal{A})$ *can be embedded into the arrangement's complement, and is a strong deformation retract of $\mathcal{M}(\mathcal{A})$.*

There is another way to look at this complex. Indeed, the poset $\mathcal{S}(\mathcal{A})$ satisfies the conditions given in [6] for a general poset to be actually the poset of cells of a regular CW-complex. Thus, $\Delta(\mathcal{S}(\mathcal{A}))$ is the barycentric subdivision of a regular CW-complex that we will call $\mathrm{Sal}(\mathcal{A})$ as well.

Remark 1.3. An explicit construction of the CW-version of the Salvetti complex is the following. Start with a geometric realization of the arrangement graph, and take it as the 1-skeleton of the CW-complex. The attaching of the higher dimensional cells $[F, C]$ is defined recursively by saying that the 1-skeleton of $[F, C]$ consists of the positive minimal paths that start at C and end at the chamber opposite to C with respect to $\mathrm{supp}(F)$; a cell $[G, K]$ is then contained in the boundary of $[F, C]$ if and only if the 1-skeleton of $[G, K]$ is a directed subgraph of the 1-skeleton of $[F, C]$ (see [67]).

Example 1.4. As an example we consider the arrangement of two lines passing through the origin of $\mathbb{R}^2$. The picture illustrates the arrangement graph and two 2-cells with their boundary.

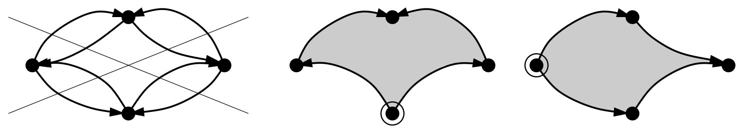

FIGURE 1. The arrangement of two lines in the plane with its arrangement graph, and two 2-cells of the cellular version of the associated Salvetti complex.

Those are the cells $[F, C]$ where F is the only codimension-2 face, namely the origin, and C is the chamber associated to the marked vertex. Note the boundary consisting of the positive minimal paths. The full complex has one such 2-cell for every chamber. △

1.4. The arrangement groupoid

A *groupoid* is a category where every arrow is invertible. This notion was first introduced by Brandt [15] as a generalization of the concept of group that he developed in his study of quadratic forms. According to [19], the use of groupoids in topology goes back to Reidemeister [65]. Let us mention also the work of Gabriel and Zisman [40], who explain and exploit the functorial relations between topological spaces, simplicial sets, and the associated groupoids. More recent textbooks exploiting groupoids in topology were written by Higgins [44] and Brown [17].

One of the classical features of groupoids is their nice covering theory, that parallels the theory of topological spaces. As this is a very classical topic, we will sketch the definitions and state the results we need; proofs and complements can be found in the elementary approach to the topic by Brown [18], while the book by Gabriel and Zisman [40] provides a more advanced treatment of the subject, together with its homological implications. For connections with homotopy of diagrams of spaces, see [19, 20]. For the basics about categories we refer to [49].

Let $\mathcal{Q}$ be a groupoid and consider an $x \in \mathrm{Ob}(\mathcal{Q})$ (we will use latin lowercase letter for objects, and Greek lowercase letters for morphisms). The set of endomorphisms $\mathrm{End}(x)$ has a natural group structure that does not depend on the choice of x; this group is called the *object group* of $\mathcal{Q}$ and will be denoted by $\pi\mathcal{Q}$ for reasons that will become clear later. The source and target object of a morphism ω will be indicated by $\mathrm{start}(\omega)$ and $\mathrm{end}(\omega)$, respectively. The *star* of the object x is the set

$$\mathrm{St}(x) := \{\omega \in \mathrm{Mor}(\mathcal{Q}) \mid \mathrm{start}(\omega) = x\}$$

of all morphisms of $\mathcal{Q}$ that start in x. The groupoid $\mathcal{Q}$ is called *connected* if for every $x, y \in \mathrm{Ob}(\mathcal{Q})$ there is a morphism ω with $x = \mathrm{start}(\omega)$ and $y = \mathrm{end}(\omega)$.

Definition 1.5. A *morphism of groupoids* is a functor

$$\rho : \mathcal{Q}' \to \mathcal{Q}$$

between groupoids. If $\mathcal{Q}$ is connected, then ρ is called a *covering* if, for every $z \in \mathrm{Ob}(\mathcal{Q}')$, the induced map

$$\rho_z : \mathrm{St}(z) \to \mathrm{St}(\rho(z))$$

is bijective. Given an $\alpha \in \mathrm{Mor}(\mathcal{Q})$ and any $z \in \rho^{-1}(\mathrm{start}(\alpha))$, the *lift of α at z* is the morphism $\rho_z^{-1}(\alpha)$, and will be written $\alpha^{\langle z \rangle}$ when the covering ρ is understood.

Example 1.6. The groupoid described in Example 1.14 is a cover of the groupoid of Example 1.11: the bijection can be checked directly. At the end of Example 1.15 we sketch the proof that the groupoid of Example 1.15 is a cover of the one defined in Example 1.12. △

If ρ is a covering of groupoids as above, the object group $\mathrm{End}(z) = \pi\mathcal{Q}'$ is mapped isomorphically by ρ to a subgroup of $\mathrm{End}(\rho(z)) = \pi\mathcal{Q}$ that is called the *characteristic group* of the covering.

The following result is classical.

Theorem 1.7 (see 9.4.3 of [18]**).** *Let $\mathcal{Q}$ be a connected groupoid, H a subgroup of $\pi\mathcal{Q}$, and choose a base object $x \in \mathrm{Ob}(\mathcal{Q})$. Consider the groupoid $\mathcal{Q}'$ defined by setting $\mathrm{Ob}(\mathcal{Q}') := \{H\omega \mid \omega \in \mathrm{St}(x)\}$ and where the morphisms between $H\omega_1$ and $H\omega_2$ correspond to morphisms α from $\mathrm{end}(\omega_1)$ to $\mathrm{end}(\omega_2)$ in $\mathcal{Q}$ such that $H\omega_1\alpha = H\omega_2$.*

The functor $\rho : \mathcal{Q}' \to \mathcal{Q}$ mapping $H\omega$ to $\mathrm{end}(\omega)$ is a covering of groupoids with characteristic group H.

Definition of the arrangement groupoid. Consider the free category on the arrangement graph (see [49, section II.4] for the definition), whose morphisms correspond to directed paths in $G(\mathcal{A})$.

Example 1.8. Take as an example the 1-dimensional arrangement given by the zero point inside the real line, that we will call $\mathcal{A}_1$. This arrangement has clearly two chambers A, B, and its arrangement graph consists of two vertices joined by two directed edges: the edge a directed from A to B, and the edge b directed from B to A (see Figure 5). The free category on it has two objects A, B, and the sets of morphisms are

$$\begin{aligned} \mathrm{Mor}(A, A) &= \{(ab)^n \mid n \in \mathbb{N}_{\geq 0}\}, \\ \mathrm{Mor}(A, B) &= \{(ab)^n a(ba)^m \mid m, n \in \mathbb{N}_{\geq 0}\} = \{(ab)^n a \mid n \in \mathbb{N}_{\geq 0}\}, \end{aligned}$$

and analogously for $\mathrm{Mor}(B, B)$ and $\mathrm{Mor}(B, A)$. △

Returning to the general situation, let R denote the smallest equivalence relation compatible with morphism composition and that identifies every two morphisms that come from positive minimal paths with the same beginning and target. We might then build the quotient category $\mathcal{G}^+ := \mathtt{Free}(G(\mathcal{A}))/R$, called the category of positive paths.

It is clear that $\mathrm{Ob}(\mathcal{G}^+(\mathcal{A})) = \mathcal{T}(\mathcal{A})$. In general, the equivalence relation is such that any two chambers C_1, C_2 determine an equivalence class of positive minimal paths starting at C_1 and ending at C_2; we will write $(C_1 \to C_2)$ for any morphism representing this class.

Example 1.9. In the previous example, the relation is empty. To see a case where it actually plays a role, let $\mathcal{A}_2$ be the arrangement of two lines considered in Example 1.4 and depicted in Figure 1 together with its arrangement graph. The vertex set of $G(\mathcal{A}_2)$ is $\{C_0, C_1, C_2, C_3\}$ (say, in counterclockwise order in Figure 1) and we may label the edges $e_{i,i\pm 1}$, where the edge $e_{i,j}$ is directed from the vertex C_i to the vertex C_j (the indexing is taken modulo 4). The set of morphisms from C_i to C_j in the free category $\mathtt{Free}(G)$ is the set of directed paths in G starting at C_i

and ending in C_j. The positive minimal paths in G are either single-edge paths or paths of length 2 of the form

$$e_{i,i+1}e_{i+1,i+2} \quad \text{or} \quad e_{i,i-1}e_{i-1,i-2}.$$

For any fixed i, the two above paths share the same start and the same target. Let us represent a directed path in G with the corresponding word in the alphabet $\{e_{i,i\pm1}\}_{i=0,\dots,3}$. The set of morphisms from C_i to C_j in $\mathcal{G}^+$ is obtained from the set of directed paths from C_i to C_j after identification of any two paths represented by words that can be transformed into one another by a sequence of substitutions of the form $e_{i,i+1}e_{i+1,i+2} \leftrightarrow e_{i,i-1}e_{i-1,i-2}$.

To describe the set of morphisms of $\mathcal{G}^+$ in this case, consider any directed path in the arrangement graph starting, say, at C_0, and let us parse it following the orientation of the edges. The first two letters of ω either describe a loop (in which case the third letter represents an edge starting at C_0) or a positive minimal path. In the latter case, two situations may occur. If the second and third letter define a directed loop in the graph, then we can apply two "substitutions" as in Figure 2 to see that this three-edges path is equivalent to one which makes a loop based at C_0, and thus ω is equivalent to a path whose third vertex is again C_0.

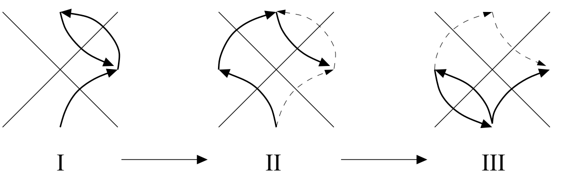

FIGURE 2

If the second and third letter define a positive minimal path, then we are already in the situation of Figure 2.II and one substitution suffices to show that ω is equivalent to a path whose third vertex is C_0. So in any case we know that ω is equivalent to a loop followed by a directed path ω', still starting at C_0 but two edges shorter than ω. By induction we see that $\mathrm{Mor}_{\mathcal{G}^+}(C_0, C_0)$ is the free commutative monoid with generators $e_{0,1}e_{1,0}$ and $e_{0,3}e_{3,0}$, and

$$\mathrm{Mor}_{\mathcal{G}^+}(C_0, C_j) = \{\alpha(C_0 \to C_j) \mid \alpha \in \mathrm{Mor}_{\mathcal{G}^+}(C_0, C_0)\}. \qquad \triangle$$

Let us again return to the general construction. We can now state the definition of the arrangement groupoid.

Definition 1.10. The *arrangement groupoid* $\mathcal{G}(\mathcal{A})$ is obtained from the category of positive paths $\mathcal{G}^+(\mathcal{A})$ by groupoid completion, i.e., adding formal inverses to every morphism.

The arrangement $\mathcal{A}$ being often understood, we will sometimes just write $\mathcal{G}$ for $\mathcal{G}(\mathcal{A})$.

Example 1.11 (Example 1.8 continued). We already described the objects and morphisms of $\mathcal{G}_1^+ := \mathcal{G}^+(\mathcal{A}_1)$ for the arrangement of one point in the real line. The associated arrangement groupoid is obtained by formally adding an element $a^{-1} \in \mathrm{Mor}(B, A)$ such that $a^{-1}a = aa^{-1} = id$ in $\mathrm{Mor}(A, A)$, and an analogous element $b^{-1} \in \mathrm{Mor}(A, B)$. Thus we have $abb^{-1}a^{-1} = id$ in $\mathrm{Mor}(A, A)$, which justifies the notation $(ab)^{-1} := b^{-1}a^{-1}$. Then, in the arrangement groupoid $\mathcal{G}_1 := \mathcal{G}(\mathcal{A}_1)$, we have

$$\mathrm{Mor}_{\mathcal{G}_1}(A, A) = \{(ab)^n \mid n \in \mathbb{Z}\} \text{ (i.e., the group } \mathbb{Z}),$$
$$\mathrm{Mor}_{\mathcal{G}_1}(A, B) = \{(ab)^n x \mid n \in \mathbb{Z},\, x = a \text{ or } b^{-1}\} \qquad \triangle$$

Example 1.12 (Example 1.9 continued). Let us now consider again the arrangemet $\mathcal{A}_2$. As in the previous example, adding formal inverses to every $e_{i,\pm 1}$ in the positive category $\mathcal{G}^+(\mathcal{A}_2)$ that we described in Example 1.9, we can for instance see that, in the arrangement groupoid $\mathcal{G}_2 := \mathcal{G}(\mathcal{A}_2)$, $\mathrm{Mor}_{\mathcal{G}_2}(C_0, C_0)$ is the free abelian group on two generators $e_{o,1}e_{1,0}$ and $e_{o,3}e_{3,0}$. $\triangle$

Remark 1.13. The arrangement groupoid was first defined by Deligne [29, (1.25)]. See also the work of Paris [61] for more on the construction. As a word of caution it has to be pointed out that in [29] this object is defined under the assumption that the arrangement is simplicial, thereby obtaining ‘by default’ some properties that are not granted in the general case, such as the faithfulness of the natural functor $\mathcal{G}^+ \to \mathcal{G}$ that turns out to be a crucial property in view of asphericity of the complement (see [29, 69, 60] and our Section 5.1). Note that our two examples indeed enjoy this property.

Coverings of the arrangement groupoid. From the definition of $\mathcal{G}(\mathcal{A})$ and from Remark 1.3 we see that indeed $\pi\mathcal{G}(\mathcal{A}) = \pi_1(\mathcal{M}(\mathcal{A}))$. So the same subgroups characterize the coverings of $\mathcal{M}(\mathcal{A})$ and the coverings of $\mathcal{G}(\mathcal{A})$.

If we apply Theorem 1.7 to the arrangement groupoid, we obtain coverings $\rho : \mathcal{G}_\rho \to \mathcal{G}$. The objects of $\mathcal{G}_\rho$ represent (right cosets of) paths on the arrangement graph. Therefore we will freely switch between the interpretation of them as objects (written with latin letters) or as morphisms in $\mathcal{G}$ (written with Greek letters). Moreover, universal covering will be denoted by a hat on the corresponding object. So $\hat{\rho}$ for the universal covering morphism and $\widehat{\mathcal{G}}$ for $\mathcal{G}_{\hat{\rho}}$.

Example 1.14. Consider the arrangement groupoid $\mathcal{G}_1$ of Examples 1.8 and 1.11. Choose A as the base point and let $\widehat{\mathcal{G}}_1$ be the groupoid given by

$$\mathrm{Ob}(\widehat{\mathcal{G}}_1) := \{v_k\}_{k\in\mathbb{Z}}, \quad \mathrm{Mor}_{\widehat{\mathcal{G}}_1}(v_i, v_j) := \{\mu_{i,j}\},$$

$$\hat{\rho} : \quad v_i \mapsto \begin{cases} A & i \text{ even} \\ B & i \text{ odd} \end{cases}$$
$$\mu_{i,j} \mapsto \begin{cases} (ab)^{q(i,j)}a^{p(i,j)} & i \text{ even, } i < j \\ (ab)^{q(i,j)+1}b^{-p(i,j)} & i \text{ even, } i > j \\ (ba)^{q(i,j)}b^{p(i,j)} & i \text{ odd, } i < j \\ (ba)^{q(i,j)+1}a^{-p(i,j)} & i \text{ odd, } i > j \end{cases}$$