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Arrangements of Hyperplanes

With 43 Figures

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To our parents

Preface

An arrangement of hyperplanes is a finite collection of codimension one affine subspaces in a finite dimensional vector space. In this book we study arrangements with methods from combinatorics, algebra, algebraic geometry, topology, and group actions. These first sentences illustrate the two aspects of our subject that attract us most. Arrangements are easily defined and may be enjoyed at levels ranging from the recreational to the expert, yet these simple objects lead to deep and beautiful results. Their study combines methods from many areas of mathematics and reveals unexpected connections.

The idea to write a book on arrangements followed three semesters of lectures on the topic at the University of Wisconsin, Madison. Louis Solomon lectured on the combinatorial aspects in the fall of 1981. Peter Orlik continued the course with the topological properties of arrangements in the spring of 1982. Hiroaki Terao visited Madison for the academic year 1982–83 and gave a course on free arrangements in the fall of 1982. The original project was to enlarge the lecture notes into a book and a long but incomplete manuscript was typed up in 1984.

The book was revived in 1988 when P. Orlik was invited to give a CBMS lecture series on arrangements. The resulting lecture notes provided a core for the present work. Also in 1988, H. Terao gave a course on arrangements at Ohio State University. Parts of his lectures are used in this book. The fresh start allowed us to take advantage of the recent TEXnical innovations of word processing and computer–aided typesetting.

P. Orlik had the opportunity to lecture on arrangements at the Swiss Seminar in Bern. He wishes to thank the mathematicians in Geneva for their hospitality and the participants of the seminar for many helpful comments. H. Terao gave a graduate course on arrangements at International Christian University in Tokyo in the fall of 1989. He wishes to thank all the participants of the course.

Time constraints forced L. Solomon to leave the completion to us. Much of the material in the book is his joint work with one or both of us, and a large part of Chapter 2 was written by him. We are grateful for permission to use his work, and for his support and friendship.

We thank W. Arvola for permission to include a modified version of his presentation of the fundamental group of the complement, P. Edelman and V. Reiner for permission to include their example, and M. Falk and R. Randell for help on many technical points. In addition, we owe thanks to C. Greene, L. Paris,

M. Salvetti, S. Yuzvinsky, T. Zaslavsky, and G. Ziegler for valuable suggestions, to V. I. Arnold for references on the M–property, and to N. Spaltenstein for references on connections with representation theory.

We thank A. B. Orlik for editing and proofreading the manuscript.

University of Wisconsin *P. Orlik and H. Terao*
February 1992

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1. Introduction

1.1 Introduction

Show that n cuts can divide a cheese into as many as $(n+1)(n^2-n+6)/6$ pieces.

Problem E 554. *Amer. Math. Monthly* **50** (1943), p. 59.
Proposed by J. L. Woodbridge, Philadelphia

Solution by Free Jamison, Pittsburgh. [*ibid* pp. 564–5] Since n straight lines can divide a plane into $(n^2+n+2)/2$ areas, the $(n+1)$st plane can be divided by the first n planes into that number of areas. For each of these areas the $(n+1)$st plane divides a piece of cheese already formed into two, and increases the total number of pieces by $(n^2+n+2)/2$. Since $(n^3+5n+6)/6$ gives the number of pieces for $n=1$ or 2, and since

$$\frac{n^3+5n+6}{6}+\frac{n^2+n+2}{2}=\frac{(n+1)^3+5(n+1)+6}{6},$$

the expression $(n^3+5n+6)/6$ holds for every n.

It is interesting to note that

(1) n points can divide a line into $1+n$ parts,

(2) n lines can divide a plane into $1+n+\binom{n}{2}$ parts,

(3) n planes can divide space into $1+n+\binom{n}{2}+\binom{n}{3}$ parts.

Editorial Note. The general formula

$$1+n+\binom{n}{2}+\binom{n}{3}+\cdots+\binom{n}{m},$$

for the case of an m-dimensional cheese, was obtained by L. Schläfli on page 39 of his great posthumous work, *Theorie der vielfachen Kontinuität* (Denkschriften der Schweizerischen naturforschenden Gesellschaft, vol. 38, 1901).

History

Such are the humble origins of our subject. In order to maximize the number of pieces, the arrangement of planes in the problem must be in "general position."

This means that any two planes have a common line, and these lines are distinct, and that any three planes have a common point, and these points are distinct. Allowing degeneracy makes the problem of counting parts much harder. In 1889, S. Roberts [191] gave a formula for the number of regions formed by an arbitrary arrangement of n lines in the plane. It is "the number of regions formed by n lines in general position" minus "the number of regions lost because of multiple points" minus "the number of regions lost because of parallels." See J. Wetzel's article [249] for a modern treatment. There is an extensive literature on partition problems in Euclidean space and projective spaces. B. Grünbaum summarized much of what was known in 1971 in [100, 101]. We quote from the introduction of his paper [100], whose title we borrowed for this book.

> ...I would like to survey the somewhat related field of *arrangements of hyperplanes*, which I expect to become increasingly popular during the next few years ...the theory of arrangements may be developed, much like topology, in rectilinear or curved versions as well as in discrete and continuous variants, and that in these developments it impinges upon many aspects of convexity, topology, and geometry which seemed to be quite unrelated.

The complement of certain hyperplanes in complex space had been studied by E. Fadell, R. Fox, and L. Neuwirth [73, 86] in connection with the braid space. The braid arrangement consists of the hyperplanes $H_{i,j} = \ker(z_i - z_j)$. Let $M = \{z \in \mathbb{C}^\ell \mid z_i \neq z_j \text{ for } i \neq j\}$ be the complement of these hyperplanes, called the pure braid space. They proved that M is a $K(\pi, 1)$ space. Let $\text{Poin}(M, t) = \sum_{k \geq 0} \text{rank} H^k(M) t^k$ be its Poincaré polynomial. In 1969, V. I. Arnold [6] proved the beautiful formula

$$\text{Poin}(M, t) = (1 + t)(1 + 2t) \cdots (1 + (\ell - 1)t) \tag{1}$$

in connection with his work on Hilbert's 13th problem. He also constructed a graded algebra A as the quotient of an exterior algebra by a homogeneous ideal, and showed that there is an isomorphism of graded algebras

$$H^*(M) \simeq A. \tag{2}$$

This gives a presentation of the cohomology ring of the pure braid space in terms of generators and relations. The study of the topological properties of the complement of an arbitrary arrangement over the complex numbers was launched by Arnold with the following remark at the end of his paper:

> Let M be the manifold obtained from $\mathbb{C}^n$ by discarding an arbitrary number of hyperplanes
>
> $$M = \{z \in \mathbb{C}^n \mid \alpha_k(z) \neq 0,\ k = 1, \ldots, N\}.$$
>
> Probably, the ring $H^*(M, \mathbb{Z})$ is torsion free and is generated by the one-dimensional classes $\omega_k = (1/2\pi i)(d\alpha_k/\alpha_k)$, a polynomial in ω_k being cohomologous to 0 in H^* only when it is zero.

E. Brieskorn [41] proved these conjectures in a 1971 Bourbaki Seminar talk. One of his results captured an essential topological feature of arrangements.

> ...une famille finie quelconque d'hyperplans affines complexes V_i, $i \in I$, dans un espace affine complexe V. Pour calculer le p-iéme groupe de cohomologie, $0 \leq p \leq n$, on considére les sousensembles maximaux $I_{p,1}, \ldots, I_{p,k_p}$ de I pour lequels on ait la propriété:
>
> $$\text{codim} \bigcap_{i \in I_{p.k}} V_i = p.$$
>
> **Lemme 3.** *Pour les complémentaires d'union d'hyperplans $Y = V - \cup_{i \in I} V_i$ et $Y_{p,k} = V - \cup_{i \in I_{p,k}} V_i$ les inclusions $i_k : Y \to Y_{p,k}$ induisent un isomorphisme:*
>
> $$H^p(Y, \mathbb{Z}) = \bigoplus_{k=1}^{k_p} H^p(Y_{p,k}, \mathbb{Z}).$$

Brieskorn also generalized Arnold's results in another direction. He replaced the symmetric group and the braid arrangement by a finite Coxeter group W and its reflection representation in a real vector space $V_{\mathbb{R}}$ of dimension ℓ. Let V be the complexification of $V_{\mathbb{R}}$. Then W acts as a reflection group in V. Let $M_W \subset V$ be the complement of the reflecting hyperplanes of W. He proved that the analog of (1) involves the exponents $m_1, \ldots m_\ell$ of W.

$$\text{Poin}(M_W, t) = (1 + m_1 t)(1 + m_2 t) \cdots (1 + m_\ell t). \tag{3}$$

Brieskorn conjectured that M_W is a $K(\pi, 1)$ space for all Coxeter groups W. He proved this for some of the groups by representing M_W as the total space of a sequence of fibrations.

In the 1971 paper quoted above, Grünbaum [100] reported the

> ...finding of a rock (or rather, unpolished gem) discovered thirty years ago by one of the lone wanderers in the wilderness of specialization. The "simplicial arrangements" which will be discussed below were first discovered by Melchior [154]; though they are a very natural notion and appear in the solutions of many problems about arrangements, they remained unnoticed.

Grünbaum listed all known simplicial arrangements in the affine and projective planes with ≤ 38 lines. It seems like poetic justice that these "unpolished gems" became the central objects in the 1972 solution of Brieskorn's conjecture by P. Deligne [61].

> **Théorème.** *Soit V un espace vectoriel réel de dimension finie, $\mathcal{M}$ un ensemble fini d'hyperplans homogénes de V, $V_{\mathbb{C}}$ le complexifié de V et $Y = V_{\mathbb{C}} - \cup_{M \in \mathcal{M}} M_{\mathbb{C}}$. On suppose que les composants connexes de $V - \cup_{M \in \mathcal{M}} M$ sont des cônes simpliciaux ouverts. Alors, Y est un $K(\pi, 1)$.*

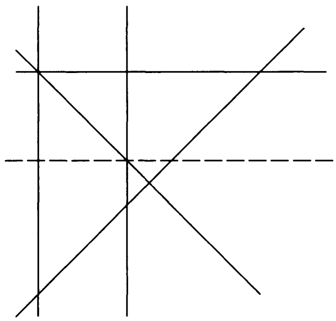

Fig. 1.1. An illustration of chamber counting

The next significant advance was made by T. Zaslavsky [256] in 1975. The title of his AMS Memoir tells it all: "*Facing up to Arrangements: Face-Count Formulas for Partitions of Space by Hyperplanes.*" He introduced the method of deletion and restriction to obtain recursion formulas for counting problems. A similar result was obtained independently by M. Las Vergnas [136]. Let $\mathcal{A}$ be an arrangement and let $H \in \mathcal{A}$ be a hyperplane. Then $\mathcal{A}' = \mathcal{A} \setminus \{H\}$ is called the deleted arrangement. The arrangement in H defined by $\mathcal{A}'' = \{K \cap H \mid K \in \mathcal{A}'\}$ is called the restricted arrangement. The triple $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ may be used to solve the problem of counting the parts of the complement of the hyperplanes of an arbitrary real arrangement. The parts are called **chambers** in modern terminology. Let $\mathcal{C}(\mathcal{A})$ be the set of chambers of $\mathcal{A}$. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of real arrangements. Then

$$|\mathcal{C}(\mathcal{A})| = |\mathcal{C}(\mathcal{A}')| + |\mathcal{C}(\mathcal{A}'')|.$$

To prove this recursion, let P be the set of those chambers in $\mathcal{C}(\mathcal{A}')$ which intersect the distinguished hyperplane H. Let Q be the set of those chambers in $\mathcal{C}(\mathcal{A}')$ which do not intersect H. Evidently $|\mathcal{C}(\mathcal{A}')| = |P| + |Q|$. The hyperplane H divides each chamber of P into two chambers of $\mathcal{C}(\mathcal{A})$ and leaves the chambers of Q unchanged. Thus $|\mathcal{C}(\mathcal{A})| = 2|P| + |Q|$. Finally, there is a bijection between P and $\mathcal{C}(\mathcal{A}'')$ given by $C \mapsto C \cap H$. Thus $|\mathcal{C}(\mathcal{A}'')| = |P|$. Figure 1.1 illustrates this in the plane. Let H be the broken line. Then $|P| = 4$ and $|Q| = 10$, so we get $|\mathcal{C}(\mathcal{A})| = 18$.

Zaslavsky defined the set $L(\mathcal{A})$ of intersections of elements of $\mathcal{A}$ and partially ordered $L(\mathcal{A})$ by reverse inclusion. He used the Möbius function of $L(\mathcal{A})$ to define the characteristic polynomial of $L(\mathcal{A})$. There is a closely related polynomial $\pi(\mathcal{A}, t)$, defined on $L(\mathcal{A})$, which we call the Poincaré polynomial. It follows from the definition that for the empty arrangement $\pi(\mathcal{A}, t) = 1$. He proved a result about the characteristic polynomial, which amounts to the following recursion for the Poincaré polynomial: $\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t)$. Since $|\mathcal{C}(\mathcal{A})|$ and $\pi(\mathcal{A}, 1)$ agree on the empty arrangement and satisfy the same recursion for deletion and restriction, this proves Zaslavsky's beautiful result:

$$|\mathcal{C}(\mathcal{A})| = \pi(\mathcal{A}, 1). \tag{4}$$

Analysis led to development in a different direction. The classical hypergeometric function $F(a, b; c; z)$ is defined by the series

$$F(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a, m)(b, m)}{(c, m)(1, m)} z^m,$$

where (a, m) denotes the factorial function

$$a(a + 1) \cdots (a + m - 1) = \frac{\Gamma(a + m)}{\Gamma(a)}.$$

The hypergeometric function satisfies the differential equation

$$z(1 - z)F'' + (c - (a + b + 1)z)F' - abF = 0,$$

and it has the Euler integral representation

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1}(1 - t)^{c-b-1}(1 - zt)^{-a} dt.$$

The function is normalized to depend on the arrangement of the three points $0, 1, z^{-1}$ in the complex line. Hypergeometric functions have been defined in several complex variables, and they have analogous integral representations. These generalizations naturally depend on arrangements of hyperplanes in affine space; see [2, 109]. Much work has been done in studying these integrals over various domains. This was the motivation for A. Hattori's 1975 investigation [108] of the homotopy type of the complement of an arrangement of complex hyperplanes in general position.

> We denote by $\mathbf{k}$ the set $\{1, 2, \ldots, k\}$. If I is a subset of $\mathbf{k}$, we denote by $|I|$ the cardinal number of I. We define the subtorus T_I of T^k by
>
> $$T_I = \{z \mid z = (z_1, \ldots, z_k) \in T^k,\ z_j = 1 \text{ for } j \notin I\}.$$
>
> The dimension of T_I is equal to $|I|$.
>
> THEOREM 1. *Let $L_1, \ldots, L_k$ be affine hyperplanes in $\mathbb{C}^n$ in general position, where $n + 1 \leq k$. Then the space $X = \mathbb{C}^n - L_1 \cup \cdots \cup L_k$ has the same homotopy type as the space*
>
> $$X_0 = \bigcup_{\substack{I \subset \mathbf{k} \\ |I| = n}} T_I.$$

This is the complex analog of the cheese cutting problem. There the number of parts in the complement of a real arrangement in general position depends only on the number of hyperplanes, but not on their location. Here the homotopy type of the complement of a complex arrangement in general position depends only on the number of hyperplanes, but not on their location.

More tools were added to the study of arrangements in 1980. P. Orlik and L. Solomon [171] used combinatorial methods to study the complement $M(\mathcal{A})$ of a