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Edited by

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Preface

Abelian Varieties are in many ways a unique class of compact complex manifolds:

- Abelian Varieties of dimension one (elliptic curves) were of major historical importance in the development of the theory of complex-analytic functions. This became a mathematical theory in its own right only after the inversion of the elliptic integral.
- The modern development of Algebraic Geometry began when Riemann introduced topological and transcendental analytic techniques. The abelian integrals used by him and his followers to understand algebraic curves have nowadays been translated into geometrical language as the canonical map of a curve to its Jacobian.
- Periods and moduli of abelian varieties have always been of interest in the development of number theory. In the creation of "arithmetical geometry" during the last two decades abelian varieties were one of the essential tools.

These three links between abelian varieties and other fields are, historically speaking, the most important. There are others, some still developing at the moment, such as the use of abelian varieties in algebraically completely integrable Hamiltonian systems.

Thus it is not surprising that the mathematical treatment of abelian varieties has concentrated on many diverging aspects, which often differ widely in their methods, aims and even language.

The Egloffstein conference was organized with the intention of bringing together mathematicians working on different aspects of the field.

As a result the talks given at the conference, as well as the contributions collected in this volume cover various aspects of present-day developments. It is the hope of the organizers that the publication will be useful for all those wishing to study ideas or problems in this large and expanding area of research.

We are indebted to the Deutsche Forschungsgemeinschaft for supporting the conference in the framework of its "Forschungsschwerpunkt Komplexe Mannigfaltigkeiten" and to the de Gruyter Verlag for publishing these proceedings. We are grateful to S. Schmiedl and R. Wetke for help with the preparation of TEX-files.

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Quadratic equations for level-3 abelian surfaces

W. Barth

0. Introduction

Let A be an abelian surface with principal polarization $\mathcal{O}_A(\Theta)$. It is well-known that the line bundle $\mathcal{O}_A(3\Theta)$ defines an embedding of A into $\mathbb{P}_8$. A theorem of G. Kempf [K] states that the ideal of A is generated by quadrics and cubics [BL, chap. 7, thm. (4.1) b]. If $A = E_1 \times E_2$ is a product of elliptic curves with product polarizations, then the embedding $E_1 \times E_2 \hookrightarrow \mathbb{P}_8$ is the product embedding $E_1 \times E_2 \hookrightarrow \mathbb{P}_2 \times \mathbb{P}_2$, with $E_i \subset \mathbb{P}_2$ a cubic curve, followed by the Segre embedding $\mathbb{P}_2 \times \mathbb{P}_2 \hookrightarrow \mathbb{P}_8$. So each quadric vanishing on the image of A in $\mathbb{P}_8$ obviously vanishes on the Segre fourfold $\mathbb{P}_2 \times \mathbb{P}_2 \subset \mathbb{P}_8$. Hence in this case cubic equations are definitely needed to define A.

The quadrics vanishing on A are well-known classically [C, AR]. It is usually assumed (cf. e.g. [vG]) that they define the surface if A is general. Coble even claims that A is the complete intersection of these quadrics [C, p. 357]. His argument seems to be that the parameters in these equations in some sense, indeed determine the surface. I cannot recognize there, however, a proof for the complete intersection property. Probably Coble did not use the notion 'Complete Intersection' in the way we do it nowadays.

In fact, the quadrics certainly do not generate the *homogeneous ideal* of the abelian surface $A \subset \mathbb{P}_4$: This follows comparing

$$h^0(\mathcal{I}_A(3)) \geq h^0(\mathcal{O}_{\mathbb{P}_8}(3)) - h^0(\mathcal{O}_A(9\Theta)) = 165 - 81 = 84$$

with the rank of the multiplication map

$$\underbrace{H^0(\mathcal{O}_{\mathbb{P}_8}(1))}_{\dim = 9} \otimes \underbrace{H^0(\mathcal{I}_A(2))}_{\dim = 9} \to \underbrace{H^0(\mathcal{I}_A(3))}_{\dim \geq 84}.$$

Now the aim of this note is to prove the

Theorem. *If $A \subset \mathbb{P}_8$ is as before, not a product of elliptic curves with product polarizations, then the quadrics vanishing on A generate the ideal-sheaf $\mathcal{I}_{A/\mathbb{P}_8}$.*

I find this fact interesting, because general theory provides quadratic equations for abelian varieties embedded with four times an ample line bundle [BL, loc. cit. thm. (4.1) a], but fails to do so with the third power of an ample bundle.

The proof presented in this note is ad hoc brute force. It consists of three steps:

- First by explicit computation it is shown that quadrics generate the ideal sheaf $\mathcal{I}_{A/\mathbb{P}_8}$ near the six odd half-periods of A.
- Then Chern class arguments are used to prove that the ideal sheaf $\mathcal{I}_A$ is generated by the quadrics everywhere *on the surface* A.
- Finally generation is shown also everywhere *off the surface* A by ad hoc counting arguments.

In some sense the proof fits into the general philosophy to prove something on abelian varieties by proving it in the half-periods and concluding it everywhere from there by topological arguments. So, perhaps the proof given here generalizes to other cases.

Notation: The base field is the field of complex numbers. ω denotes the cube root of unity $e^{2\pi i/3}$.

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1. The situation

Let A be a principally polarized abelian surface with a level-3 structure. This is defined by a principal polarization $\mathcal{O}_A(\Theta)$ on A together with an action of the Heisenberg group $H_{3,3}$,

$$1 \to \mathbb{Z}_3 \to H_{3,3} \to (\mathbb{Z}_3)^4 \to 1,$$

on $\mathcal{O}_A(3\Theta)$. The induced action of $H_{3,3}$ on the space $H^0(\mathcal{O}_A(3\Theta))$ is isomorphic with the following nine-dimensional representation:

Put $K := \mathbb{Z}_3 \times \mathbb{Z}_3$, $V := \{ \text{ maps } f : K \to \mathbb{C}\}$, let K act on V by translation

$$K \ni \sigma : f(\alpha) \mapsto f(\alpha - \sigma),$$

and the character group K^* by multiplication

$$K^* \ni \tau : f(\alpha) \mapsto < \tau, \alpha > \cdot f(\alpha).$$

The actions of K and K^* generate the representation of $H_{3,3}$ in question. (Sometimes it is called Schroedinger representation.) The *extended Heisenberg group*

$EH_{3,3}$ is the extension of the group $H_{3,3}$ by the involution ι acting on V by

$$\iota : f(\alpha) \mapsto f(-\alpha).$$

We assume Θ symmetric, i.e. $\iota(\Theta) = \Theta$ for the involution $\iota : a \mapsto -a$ of A. Then ι lifts to an involution on $\mathcal{O}_A(3\Theta)$. A *symmetric theta-structure* is an isomorphism of $EH_{3,3}$ with the group generated by $H_{3,3}$ and ι acting on $\mathcal{O}_A(3\Theta)$. Each level-3 structure is induced by a unique symmetric theta-structure [BL, chap. 6, thm. (9.5)].

By Lefschetz's classical theorem [BL chap. 4, thm. (5.1)] the bundle $\mathcal{O}_A(3\Theta)$ is very ample and defines some $EH_{3,3}$-equivariant embedding of A into $\mathbb{P}_8 = \mathbb{P}(V)$. We denote by $\mathbb{P}H_{3,3} = H_{3,3}/\mathbb{Z}_3$ the (abelian) image of $H_{3,3}$ in the projective group $\mathbb{P}GL(V) = \mathbb{P}GL(9, \mathbb{C})$.

Let me call $X_\alpha \in V, \alpha \in K$, the delta function with support α. We view the $X_\alpha, \alpha \in K$, as homogeneous coordinates on $\mathbb{P}_8$. The representation of $EH_{3,3}$ above is generated by the transformations

$$\begin{array}{llll} \sigma_1 : & X_{k,l} & \mapsto & X_{k-1,l} \\ \tau_1 : & X_{k,l} & \mapsto & \omega^k X_{k,l} \\ \iota : & X_{k,l} & \mapsto & X_{-k,-l}. \end{array} \qquad \begin{array}{llll} \sigma_2 : & X_{k,l} & \mapsto & X_{k,l-1} \\ \tau_2 : & X_{k,l} & \mapsto & \omega^l X_{k,l} \\ \end{array}$$

We need explicitly the subspaces of fixed points for the subgroups of $\mathbb{P}H_{3,3}$:

group	subspace	proj. dim.	eigenvalue(s)
$\mathbb{Z}_3 =< \tau_1 >$	$X_{k,l} = 0$ for $k \neq 0$	2	1
	$X_{k,l} = 0$ for $k \neq 1$	2	ω
	$X_{k,l} = 0$ for $k \neq 2$	2	ω^2
$\mathbb{Z}_3 \times \mathbb{Z}_3 =< \tau_1, \tau_2 >$	$\mathbb{C} \cdot e_{m,n} : X_{k,l} = \delta_{kl}^{mn}$	0	ω^m, ω^n
$\mathbb{Z}_3 \times \mathbb{Z}_3 =< \tau_1, \sigma_1 >$	$\emptyset$		
$\mathbb{Z}_2 =< \iota >$	$\mathbb{P}_4^+ = \{X_{k,l} = X_{-k,-l}\}$	4	1
	$\mathbb{P}_3^- = \{X_{k,l} = -X_{-k,-l}\}$	3	-1

By abuse of language we call a subgroup (like $< \tau_1, \tau_2 >$) of $\mathbb{Z}_3 \times \mathbb{Z}_3$ *isotropic* if its pre-image in $H_{3,3}$ is commutative, and *anisotropic* if this is not the case (as for $< \sigma_1, \tau_1 >$).

Following [BII and C, p. 355] we use

symmetric	antisymmetric
$Y_0 = X_{0,0}$	
$Y_1 = \frac{1}{2}(X_{0,1} + X_{0,2})$	$Z_1 = \frac{1}{2}(X_{0,1} - X_{0,2})$
$Y_2 = \frac{1}{2}(X_{1,0} + X_{2,0})$	$Z_2 = \frac{1}{2}(X_{1,0} - X_{2,0})$
$Y_3 = \frac{1}{2}(X_{1,1} + X_{2,2})$	$Z_3 = \frac{1}{2}(X_{1,1} - X_{2,2})$
$Y_4 = \frac{1}{2}(X_{1,2} + X_{2,1})$	$Z_4 = \frac{1}{2}(X_{1,2} - X_{2,1})$

coordinates. In these coordinates

$$P_3^- = \{Y_0 = \cdots = Y_4 = 0\}, \qquad \mathbb{P}_4^+ = \{Z_1 = \cdots = Z_4 = 0\}.$$

There is a huge group $N \subset SL(9,\mathbb{C})$ normalizing the $H_{3,3}$-action. It fits into an exact sequence

$$1 \to H_{3,3} \to N \to Sp(4,\mathbb{F}_3) \to 1.$$

This can be seen as in [HM, p. 65]. The proof implies that $N^\iota \subset N$, the centralizer of ι in N maps onto $Sp(4,\mathbb{F}_3)$.

So the centralizer N^ι is an extension

$$1 \to \mathbb{Z}_3 \to N^\iota \to Sp(4,\mathbb{F}_3) \to 1$$

acting

on	as the group	of order
$\mathbb{P}_8$ and $\mathbb{P}_3^-$	$Sp(4,\mathbb{F}_3)$	51 840
$\mathbb{P}_4^+$	$PSp(4,\mathbb{F}_3)$	25 920.

On $\mathbb{P}_3^-$ the group N^ι acts transitively on the points and lines of *Witting's configuration* [B III, p. 318]:

40 points	$(1:0:0:0),(0:1:0:0),(0:0:1:0),(0:0:0:1)$ $(0:\omega^k:\omega^l:\omega^m),(\omega^k:0:\omega^l:-\omega^m),$ $(\omega^k:-\omega^l:0:\omega^m),(\omega^k:\omega^l:-\omega^m:0)$
45 line pairs	e.g. $Z_1 = Z_3 - Z_4 = 0$ and $Z_2 = Z_3 + Z_4 = 0$
40 planes	$Z_k = 0,$ $\omega^k Z_2 + \omega^l Z_3 + \omega^m Z_4 = 0, \quad \omega^k Z_1 + \omega^l Z_3 - \omega^m Z_4 = 0,$ $\omega^k Z_1 - \omega^l Z_2 + \omega^m Z_4 = 0, \quad \omega^k Z_1 + \omega^l Z_2 - \omega^m Z_3 = 0.$

We shall use explicitly the 36 Witting lines lying in coordinate planes:

$$Z_k = Z_m \pm \omega^i Z_n = 0, \quad \text{with} \quad \{k,l,m,n\} = \{1,2,3,4\} \quad \text{and} \quad i = 0,1,2,$$

where the $\pm$-signs are chosen according to the following pattern

$$\begin{array}{ccl} & & (k,l) \\ + & : & (2,1),(2,3),(3,1),(3,4),(4,1),(4,2) \\ - & : & (1,2),(1,3),(1,4),(2,4),(3,2),(4,3). \end{array}$$

These lines are obtained from $Z_1 = Z_3 - Z_4 = 0$ by applying the symmetries C, D and S_2 of [B III, p. 318].

The group N permutes embeddings of the same surface A differring by the level-3-structure.

(1.1) *The group N permutes transitively the fixed point sets for subgroups $\mathbb{Z}_3$, isotropic $\mathbb{Z}_3 \times \mathbb{Z}_3$ and anisotropic $\mathbb{Z}_3 \times \mathbb{Z}_3 \subset \mathbb{P}H_{3,3}$.*

Proof. The group $Sp(4, \mathbb{F}_3)$ operates transitively on the 40 lines, 90 isotropic planes, and 40 anisotropic planes in $\mathbb{F}_3^4$. □

(1.2) *For each nontrivial $\tau \in \mathbb{P}H_{3,3}$ the intersection $\mathbb{P}_4^+ \cap \tau(\mathbb{P}_4^+)$ is a line in a fix-plane for τ.*

Proof. By N^ι-equivariance we may assume $\tau = \tau_1$. Then

$$\mathbb{P}_4^+ \cap \tau(\mathbb{P}_4^+) = \{X_{0,1} - X_{0,2} = X_{1,0} - X_{2,0} = X_{1,1} - X_{2,2} = X_{1,2} - X_{2,1} = \omega X_{1,0} - \omega^2 X_{2,0} = \omega X_{1,1} - \omega^2 X_{2,2} = \omega X_{1,2} - \omega^2 X_{2,1} = 0\}$$

lies on the plane $X_{1,k} = X_{2,k} = 0$, $k = 0, 1, 2$. □

2. The product case

An elliptic curve E with origin o_E maps Heisenberg-equivariantly under the linear system $|3 \cdot o_E|$ into $\mathbb{P}_2$ as a plane cubic with equation

$$x_0^3 + x_1^3 + x_2^3 + \lambda x_0 x_1 x_2 = 0 \qquad \text{(Hesse normal form)}.$$

A product $A = E_1 \times E_2$ maps Heisenberg-equivariantly under the linear system $|3(E_1 \times o_{E_2} + o_{E_1} \times E_2)|$ onto the surface

$$X_{i,j} = x_i \cdot y_j, \quad x_0^3 + x_1^3 + x_2^3 + \lambda_1 x_0 x_1 x_2 = y_0^3 + y_1^3 + y_2^3 + \lambda_2 y_0 y_1 y_2 = 0$$

in the Segre four-fold $\mathbb{P}_2 \times \mathbb{P}_2 \subset \mathbb{P}_8$.

This Segre four-fold is the image of the map $X_{i,j} = x_i y_j$. It meets the solid $\mathbb{P}_3^-$ in the locus of points $(Z_1 : Z_2 : Z_3 : Z_4)$,

$$\begin{array}{rclcrcl} Z_1 &=& \frac{1}{2} x_0(y_1 - y_2) & & Z_2 &=& \frac{1}{2}(x_1 - x_2) y_0 \\ Z_3 &=& \frac{1}{2}(x_1 y_1 - x_2 y_2) & & Z_4 &=& \frac{1}{2}(x_1 y_2 - x_2 y_1) \end{array}$$

satisfying

$$\begin{array}{rclcl} Y_0 &=& x_0 y_0 &=& 0, \\ 2Y_1 &=& x_0(y_1 + y_2) &=& 0, \qquad 2Y_2 = (x_1 + x_2) y_0 = 0, \\ 2Y_3 &=& x_1 y_1 + x_2 y_2 &=& 0, \qquad 2Y_4 = x_1 y_2 + x_2 y_1 = 0. \end{array}$$

This means

$$\begin{array}{lll} x_0 = y_0 = 0 & \Rightarrow & Z_1 = Z_2 = 0 \\ & & \text{and } x_1 + x_2 = y_1 - y_2 = 0 \\ & & \text{or } \quad x_1 - x_2 = y_1 + y_2 = 0 \\ x_0 = 0, y_0 \neq 0 & \Rightarrow & x_1 + x_2 = y_1 - y_2 = 0, \text{ i.e. } Z_1 = Z_3 - Z_4 = 0 \\ x_0 \neq 0, y_0 = 0 & \Rightarrow & x_1 - x_2 = y_1 + y_2 = 0, \text{ i.e. } Z_2 = Z_3 + Z_4 = 0. \end{array}$$

So the intersection is the Witting line pair

$$Z_1 = Z_3 - Z_4 = 0 \quad \text{and} \quad Z_2 = Z_3 + Z_4 = 0.$$

(2.1) *If an abelian surface $A \subset \mathbb{P}_8$ as in section 1 intersects this line pair, then A is a product surface $E_1 \times E_2 \subset \mathbb{P}_2 \times \mathbb{P}_2$ on the Segre four-fold.*

Proof. Assume the intersection $I := A \cap \mathbb{P}_2 \times \mathbb{P}_2$ is not empty. Clearly it is $EH_{3,3}$-invariant and it is cut out on A by quadrics. If $\dim(I) = 0$, then I is contained in an intersection of two different curves in $|\mathcal{O}_A(6\Theta)|$, hence $|I| \leq 72$. But I consists of $H_{3,3}$-orbits of length ≥ 81, a contradiction. So I will contain a $H_{3,3}$-invariant curve C_I. Again C_I is contained in a curve from $|\mathcal{O}_A(6\Theta)|$, hence $C_I^2 \leq 72$. But by $H_{3,3}$-invariance C_I descends to $A/H_{3,3}$ and its self-intersection should be divisible by 81, again a contradiction. This implies $A \subset \mathbb{P}_2 \times \mathbb{P}_2$.

Let the homology class of A in $\mathbb{P}_2 \times \mathbb{P}_2$ be $[A] = a[\mathbb{P}_2 \times pt] + b[\mathbb{P}_1 \times \mathbb{P}_1] + c[pt \times \mathbb{P}_2]$. The degree of A is

$$a + 2b + c = 18.$$

We have to show $a = c = 0$ and $b = 9$. We compute the the self-intersection of $A \subset \mathbb{P}_2 \times \mathbb{P}_2$ in two ways:

$$\begin{array}{rcl} (A.A)_{\mathbb{P}_2 \times \mathbb{P}_2} & = & 2ac + b^2, \\ c_2(N_{A/\mathbb{P}_2 \times \mathbb{P}_2}) & = & [(1 + 3[\mathbb{P}_2 \times \mathbb{P}_1] + 3[\mathbb{P}_2 \times pt]) \cdot \\ & & \qquad (1 + 3[\mathbb{P}_1 \times \mathbb{P}_2] + 3[pt \times \mathbb{P}_2])]_2 \quad |A \\ & = & (3[\mathbb{P}_2 \times pt] + 9[\mathbb{P}_1 \times \mathbb{P}_1] + 3[pt \times \mathbb{P}_2])|A \\ & = & 3(a + c) + 9b. \end{array}$$

This implies

$$b(b - 9) = 3(a + c) - 2ac.$$

If $c = 0$, then $a = 18 - 2b$ and $b(b - 3) = 54$ implies $b = 9$ and $a = 0$. So we have to exclude the case $a > 0$ and $c > 0$. In this case however, the projections $A \to \mathbb{P}_2$ on both factors are surjective. Since A contains no exceptional curve, these projections are finite. The Hesse-pencils on both planes $\mathbb{P}_2$ therefore pull back to A as pencils with a finite set of base points. This implies $A \cap E_1 \times E_2$ is finite of degree $[A].9[\mathbb{P}_1 \times \mathbb{P}_1] = 9b$ for the general product surface $E_1 \times E_2$. Since the intersection consists of $H_{3,3}$-orbits, we find $b = 9$ and arrive at $a = 0$. □

Using the N^ι-action, statement (2.1) is generalized to all Witting lines:

(2.1′) *If an abelian surface $A \subset \mathbb{P}_8$ as in section 1 meets a Witting line on $\mathbb{P}_3^-$, then A is a product with Θ the product polarization.*

3. The quadratic equations

Assume from now on that $A = A_{3,3} \hookrightarrow \mathbb{P}_8$ as above is not a product surface.

By Sekiguchi and Koizumi [S] restriction

$$\begin{array}{ccc} H^0(\mathcal{O}_{P_8}(2)) & \to & H^0(\mathcal{O}_A(2)) \\ \dim = 45 & & \dim = 36 \end{array}$$

is surjective, so it has a nine-dimensional kernel $H^0(\mathcal{I}_A(2))$ of quadrics vanishing on A. By equivariance this kernel is invariant under the action of $EH_{3,3}$ on $H^0(\mathcal{O}_{\mathbb{P}_8}(2))$.

The space $H^0(\mathcal{O}_{\mathbb{P}_8}(2))$ of quadrics decomposes under $EH_{3,3}$ into five isomorphic representations given by the columns of the matrix

$$\begin{array}{ccccc}
X_{0,0}^2 & X_{0,1}X_{0,2} & X_{1,0}X_{2,0} & X_{1,1}X_{2,2} & X_{1,2}X_{2,1} \\
X_{0,1}^2 & X_{0,2}X_{0,0} & X_{1,1}X_{2,1} & X_{1,2}X_{2,0} & X_{1,0}X_{2,2} \\
X_{0,2}^2 & X_{0,0}X_{0,1} & X_{1,2}X_{2,2} & X_{1,0}X_{2,1} & X_{1,1}X_{2,0} \\
X_{1,0}^2 & X_{1,1}X_{1,2} & X_{2,0}X_{0,0} & X_{2,1}X_{0,2} & X_{2,2}X_{0,1} \\
X_{1,1}^2 & X_{1,2}X_{1,0} & X_{2,1}X_{0,1} & X_{2,2}X_{0,0} & X_{2,0}X_{0,2} \\
X_{1,2}^2 & X_{1,0}X_{1,1} & X_{2,2}X_{0,2} & X_{2,0}X_{0,1} & X_{2,1}X_{0,0} \\
X_{2,0}^2 & X_{2,1}X_{2,2} & X_{0,0}X_{1,0} & X_{0,1}X_{1,2} & X_{0,2}X_{1,1} \\
X_{2,1}^2 & X_{2,2}X_{2,0} & X_{0,1}X_{1,1} & X_{0,2}X_{1,0} & X_{0,0}X_{1,2} \\
X_{2,2}^2 & X_{2,0}X_{2,1} & X_{0,2}X_{1,2} & X_{0,0}X_{1,1} & X_{0,1}X_{1,0}
\end{array}$$

Each parameter $r = (r_0 : \ldots : r_4) \in \mathbb{P}_4$ determines one nine-dimensional subrepresentation. The action of N on $\mathcal{O}_{P_8}(1)$ induces an action on this space of quadrics and an action of $N/H_{3,3}$ on the parameter space $\mathbb{P}_4$.

In (anti-) symmetric coordinates $Y_0, \ldots, Y_4$ and $Z_1, \ldots, Z_4$ the quadrics take the form [C, p. 356–357]

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \end{pmatrix} = \left(\begin{pmatrix} Y_0^2 & Y_1^2 & Y_2^2 & Y_3^2 & Y_4^2 \\ Y_1^2 & Y_0Y_1 & Y_3Y_4 & Y_2Y_4 & Y_2Y_3 \\ Y_2^2 & Y_3Y_4 & Y_0Y_2 & Y_1Y_4 & Y_1Y_3 \\ Y_3^2 & Y_2Y_4 & Y_1Y_4 & Y_0Y_3 & Y_1Y_2 \\ Y_4^2 & Y_2Y_3 & Y_1Y_3 & Y_1Y_2 & Y_0Y_4 \end{pmatrix} + \right.$$

$$\left. + \begin{pmatrix} 0 & -Z_1^2 & -Z_2^2 & -Z_3^2 & -Z_4^2 \\ Z_1^2 & 0 & -Z_3Z_4 & -Z_2Z_4 & -Z_2Z_3 \\ Z_2^2 & Z_3Z_4 & 0 & Z_1Z_4 & -Z_1Z_3 \\ Z_3^2 & Z_2Z_4 & -Z_1Z_4 & 0 & Z_1Z_2 \\ Z_4^2 & Z_2Z_3 & Z_1Z_3 & -Z_1Z_2 & 0 \end{pmatrix} \right) \cdot r$$

$$\begin{pmatrix} Q_6 \\ Q_7 \\ Q_8 \\ Q_9 \end{pmatrix} =$$

$$\begin{pmatrix} -r_1Z_1 & 2r_0Z_1 & r_3Z_4 - r_4Z_3 & r_4Z_2 - r_2Z_4 & r_2Z_3 - r_3Z_2 \\ -r_2Z_2 & -r_3Z_4 - r_4Z_3 & 2r_0Z_2 & r_1Z_4 + r_4Z_1 & r_1Z_3 - r_3Z_1 \\ -r_3Z_3 & -r_2Z_4 - r_4Z_2 & r_1Z_4 - r_4Z_1 & 2r_0Z_3 & r_1Z_2 + r_2Z_1 \\ -r_4Z_4 & -r_2Z_3 - r_3Z_2 & r_1Z_3 + r_3Z_1 & r_1Z_2 - r_2Z_1 & 2r_0Z_4 \end{pmatrix} \cdot \begin{pmatrix} Y_0 \\ Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix}.$$

On $\mathbb{P}_3^-$ these quadrics restrict as

$$\begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \end{pmatrix} = \begin{pmatrix} 0 & -Z_1^2 & -Z_2^2 & -Z_3^2 & -Z_4^2 \\ Z_1^2 & 0 & -Z_3Z_4 & -Z_2Z_4 & -Z_2Z_3 \\ Z_2^2 & Z_3Z_4 & 0 & Z_1Z_4 & -Z_1Z_3 \\ Z_3^2 & Z_2Z_4 & -Z_1Z_4 & 0 & Z_1Z_2 \\ Z_4^2 & Z_2Z_3 & Z_1Z_3 & -Z_1Z_2 & 0 \end{pmatrix} \cdot \begin{pmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{pmatrix} = (q_{ij}(Z)) \cdot r.$$

We notice explicitely the *quadratic cone*

$$Q^+ := \sum_0^4 r_i Q_{i+1}$$

with top $\mathbb{P}_3^-$.

(3.1) *The quadric $Q^+ \cap \mathbb{P}_4^+$ is smooth unless $r \in \mathbb{P}_4$ belongs to an N^ι-invariant threefold of degree ten.*

Proof. In coordinates $Y_0, \ldots, Y_4$ the quadric is given by the matrix

$$\begin{pmatrix} r_0^2 & r_1^2/2 & r_2^2/2 & r_3^2/2 & r_4^2/2 \\ r_1^2/2 & 2r_0r_1 & r_3r_4 & r_2r_4 & r_2r_3 \\ r_2^2/2 & r_3r_4 & 2r_0r_2 & r_1r_4 & r_1r_3 \\ r_3^2/2 & r_2r_4 & r_1r_4 & 2r_0r_3 & r_1r_2 \\ r_4^2/2 & r_2r_3 & r_1r_3 & r_1r_2 & 2r_0r_4 \end{pmatrix}.$$

The determinant is an invariant of degree 10 for N^ι acting on $\mathbb{P}_4^+$. It does not vanish identically, as one checks e.g. taking $r_1 = r_2 = 0$. □

The system of five quadrics $q_1, \ldots, q_5$ defines a *Steiner map* $\mathbb{P}_3 \to \mathbb{P}_4$ everywhere on $\mathbb{P}_3$, where the alternating matrix (q_{ij}) has rank four. It maps $z \in \mathbb{P}_3$ to the kernel $r(z)$ of the matrix $(q_{ij}(z))$. We need the base points of this map:

(3.2) (C, p. 358) *The rank of the alternating* 5×5*-matrix* $q_{ij}(Z)$ *drops down to two precisely in the* 40 *Witting-points.*

Proof. One has to compute the five 4×4-minors in the last four rows of the matrix

(q_{ij}). They are

$$\begin{array}{rclrcl} D_1 & = & 3 \cdot Z_1 Z_2 Z_3 Z_4, & & & \\ D_2 & = & Z_1 \cdot (Z_2^3 + Z_3^3 + Z_4^3), & D_3 & = & Z_2 \cdot (Z_1^3 + Z_3^3 - Z_4^3), \\ D_4 & = & Z_3 \cdot (-Z_1^3 + Z_2^3 - Z_4^3), & D_5 & = & Z_4 \cdot (Z_1^3 + Z_2^3 - Z_3^3) \end{array}$$

up to a common factor $3 \cdot Z_1 Z_2 Z_3 Z_4$.

On the plane $Z_1 = 0$ the rank drops if and only if $Z_2 = Z_3^3 = Z_4^3$, i.e., if z is a Witting point $(0 : \omega^k : \omega^l : \omega^m)$. And under N^ι all four coordinate planes $Z_k = 0$ are equivalent. □

(3.3) *The Steiner map $r : \mathbb{P}_3^- \to \mathbb{P}_4$ is equivariant for the N^ι-action. It*

a) *blows up the* 40 *Witting points,*

b) *blows down the* 90 *Witting lines,*

c) *has differential not of maximal rank on the exceptional planes replacing the blown up points.*

Proof. a) has been checked above already. Each Witting line contains four Witting points. The map r is defined by a linear system of quartics. This proves b). To prove c) we blow up e.g. the point $(1 : 0 : 0 : 0)$ putting $Z_1 = 1$ and $Z_3 = uZ_2, Z_4 = vZ_2$. After removing the common factor Z_2 the map r is given in the Z_2, u, v-chart by

$$\begin{array}{rclrcl} D_1 & = & 3 \cdot uv \cdot Z_2^2, & & & \\ D_2 & = & (1 + u^3 + v^3) Z_2^2, & D_3 & = & 1 + (u^3 - v^3) \cdot Z_2^2, \\ D_4 & = & -u + u(1 - v^3) \cdot Z_2^2, & D_5 & = & v + v(1 - u^3) \cdot Z_2^2. \end{array}$$

The matrix of derivatives is easily seen to have rank two on the surface $Z_2 = 0$. □

(3.4) *If the nine quadrics $Q_1, \ldots, Q_9$ vanish on A, then they will not vanish in a coordinate vertex e_{kl}.*

Proof. Assume that the nine quadrics vanish in some coordinate point e_{kl}. This is equivalent with $r_0 = 0$.

Let $P = (p_1 : p_2 : p_3 : p_4) \in A \cap \mathbb{P}_3^-$ be one of the six half-periods in A. By $r_0 = 0$ the lower right-hand minor $(D_1)^2$ vanishes there, and $p_1 \cdot p_2 \cdot p_3 \cdot p_4 = 0$. So P lies on a coordinate plane $Z_k = 0$. On this plane the conditions $q_1 = \cdots = q_5 = 0$ imply $r_k Z_l Z_m = 0$ for all $1 \leq l, m \leq 4, l, m \neq k$. If $r_k = 0$, then P would be a coordinate point, in conflict with (2.1'). So at least one parameter $r_k, k > 0$ vanishes.

If $r_k = 0$ for one $k > 0$ only, then the six odd half-periods in $A \cap \mathbb{P}_3^-$ lie in the plane $Z_k = 0$ in the intersection of the two smooth conics

$$\sum_{0<l\neq k} r_l Z_l^2 = \sum_{0<l,m,n\neq k} \pm r_k Z_l Z_n = 0,$$

a contradiction. So another parameter $r_l, l \neq 0, k$ will vanish.

Four parameters r_i cannot vanish, because then again all points $P \in A \cap \mathbb{P}_3^-$ would be coordinate points. So we find that precisely three parameters $r_0, r_k, r_l, k \neq l > 0$ will vanish, and that the six points $P \in A \cap \mathbb{P}_3^-$ lie on the two planes $Z_k Z_l = 0$.

Not all six odd half-periods P can lie on the line $Z_k = Z_l = 0$, because there the quadric $r_m Z_m^2 + r_n Z_n^2$ cuts out two distinct points only. So for at least four points we have $Z_k \neq 0$ or $Z_l \neq 0$. Assume $Z_k = 0$ and $Z_l \neq 0$ for at least two of these points. Then they lie on the plane $Z_k = 0$ in the intersection of the line pair

$$r_m Z_m^2 + r_n Z_n^2 = 0$$

with the line

$$r_m Z_n \pm r_n Z_m = 0 \text{ where } \begin{cases} + \text{ for } (k,l) = (1,2),(1,3),(1,4),(2,4),(3,2),(4,3) \\ - \text{ for } (k,l) = (2,1),(2,3),(3,1),(3,4),(4,1),(4,2). \end{cases}$$

These three lines are concurrent, so the third line must coincide with one line from the line pair, hence

$$\det \begin{pmatrix} \sqrt{r_m} & i\sqrt{r_n} \\ \pm r_n & r_m \end{pmatrix} = r_m^{3/2} \mp i r_n^{3/2} = 0.$$

This implies

$$r_m^3 = -r_n^3, \qquad r_m + \omega^j r_n = 0 \text{ for } j = 0, 1, \text{ or } 2.$$

The line

$$Z_k = Z_m \mp \omega^j Z_n = 0$$

by section 1 however is a Witting line and cannot contain points $P \in A \cap \mathbb{P}_3^-$ by (2.1′). □

(3.5) *If the quadrics $Q_1, \ldots, Q_9$ vanish on A, then they will not vanish in any fixed point for a subgroup $\mathbb{Z}_3 \subset \mathbb{P}H_{3,3}$.*

Proof. E.g. on the plane $X_{1k} = X_{2k} = 0$, $k = 0, 1, 2$ the nine quadrics $Q_1, \ldots, Q_9$ cut out the locus

$$r_0 X_{00}^2 + r_1 X_{01} X_{02} = r_0 X_{01}^2 + r_1 X_{00} X_{02} = r_0 X_{02}^2 + r_1 X_{00} X_{01} = 0.$$

Since we may assume $r_0 \neq 0$ by (3.4), we find

$$X_{00}(X_{01}^3 - X_{02}^3) = X_{01}(X_{02}^3 - X_{00}^3) = X_{02}(X_{00}^3 - X_{01}^3) = 0.$$

The vanishing of one coordinate implies the vanishing of a second one, so that

$$X_{01} = \omega^k X_{00}, \quad X_{02} = \omega^l X_{00}, \quad r_0 + \omega^m r_1 = 0 \text{ for } m = k + l.$$

On $\mathbb{P}_3^-$ the quadrics cut out a locus, which except for Witting points is contained in the surface

$$\begin{aligned} -D_1 + \omega^m D_2 &= \omega^m Z_1(-3\omega^{-m} Z_2 Z_3 Z_4 + Z_2^3 + Z_3^3 + Z_4^3) \\ &= \omega^m Z_1(\omega^{-m} Z_2 + Z_3 + Z_4)(\omega^{-m} Z_2 + \omega Z_3 + \omega^2 Z_4) \cdot \\ &\qquad (\omega^{-m} Z_2 + \omega^2 Z_3 + \omega Z_4) \\ &= 0. \end{aligned}$$

For a point of A on the plane $Z_1 = 0$ we find

$$\omega^m Z_2^2 - Z_3 Z_4 = \omega^m Z_3^2 - Z_2 Z_4 = \omega^m Z_4^2 - Z_2 Z_3 = 0.$$

This is the Witting point $(0 : 1 : \omega^k : \omega^l)$ and cannot belong to A. The other three planes are transforms of the plane $Z_1 = 0$ under N^ι [B III, p. 318].

On the planes $X_{0k} = X_{1k} = 0$ and $X_{0k} = X_{2k} = 0$ the argument is similar. All noncentral subgroups $\mathbb{Z}_3 \subset H_{33}$ are permuted transitively by N, so it suffices to prove the assertion for the subgroup generated by τ_1 as we did. □

(3.6) *If the quadrics $Q_1, \dots, Q_9$ vanish on A, then they cut out on $\mathbb{P}_3^-$ precisely the six odd half-periods $e_1, \dots, e_6 \in A \cap \mathbb{P}_3^-$.*

Proof. It is well-known [C, p. 358] that the map $r : \mathbb{P}_3^- \dashrightarrow J_4$ has degree six. An additional point of intersection in $q_1(r) = \cdots = q_5(r) = 0$ then would be a Witting point. Under N^ι all of them are equivalent with $z = (1 : 0 : 0 : 0)$, which belongs to the plane $X_{kl} = 0$ for $k \neq 0$. By (3.5) the quadrics will not vanish there. □

(3.7) *If the quadrics $Q_1, \dots, Q_9$ vanish on A, then the quadric cone Q^+ cuts out a smooth quadric on $\mathbb{P}_4^+$.*

Proof. By (3.1) the quadric $Q^+ \cap \mathbb{P}_4^+$ degenerates only if r belongs to some threefold of degree 10, invariant under N^ι. There are just two basic invariants of degree 10 [B II, p. 208], namely

$$J_4 \cdot J_6 \text{ and } J_{10}.$$

All these invariants cut out on $J_4 = 0$ the union of 40 *Haupt*-planes [B II, p. 192]. They are under the group conjugated to the plane $r_0 = r_1 = 0$. The assertion follows from (3.4). □

4. Generation in the odd half-periods

The embedded surface $A \subset \mathbb{P}_8$ meets $\mathbb{P}_3^-$ in the images of its six odd half-periods $e = e_1, \dots, e_6$. The aim of this section is to show

(4.1) *The nine differentials $dQ_1(e), \dots, dQ_9(e)$ span a subspace in $T_e^*(\mathbb{P}_8)$ of dimension six.*

This assertion follows from

(4.1, on) *The five differentials $dq_1(e), \dots, dq_5(e)$ span a subspace in $T_e^*(\mathbb{P}_3^-)$ of dimension three.*

(4.1, off) *Suppose $\mathbb{P}_5(e)$ is the join of e with $\mathbb{P}_4^+$. The nine differentials $dQ_k(e)|T_e^*(\mathbb{P}_5(e))$ span in $T_e^*(\mathbb{P}_5(e))$ a subspace of the same dimension as the restrictions $dQ_k(e)|T_e^*(\mathbb{P}_3^-)$.*